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Revisitation of the product of two orthogonal projectors

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A B S T R A C T

Several results involving a product of two orthogonal projectors (i.e., Hermitian idempotent matrices) are established by exploring a representation of the product as a partitioned matrix. These results concern, for instance, rank, trace, range, null space, generalized inverses, and spectral properties of the product and its various functions. Particular attention is paid to the conditions equivalent to the requirement that the product of two orthogonal projectors is an orthogonal projector itself, and these characterizations refer to such known classes of matrices as Hermitian, involutory, normal, star-dagger, unitary as well as partial isometries and semi-orthogonal projectors. Moreover, some results dealing with the notions of parallel sum and spectral norm are obtained. The variety of problems considered shows that the approach utilized in the paper provides a powerful tool of wide applicability.

1. Introduction

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. The symbols $\mathbf{K}^*$, $\mathscr{R}(\mathbf{K})$, $\mathscr{N}(\mathbf{K})$, and $\text{rk}(\mathbf{K})$ will stand for the conjugate transpose, column space, null space, and rank of $\mathbf{K} \in \mathbb{C}_{m,n}$, respectively. Moreover, $\mathbf{I}_n$ will be the identity matrix of order n, whereas $\text{tr}(\mathbf{K})$ and $\lambda_j(\mathbf{K}), j = 1, \ldots, n$, will mean the trace and jth eigenvalue of $\mathbf{K} \in \mathbb{C}_{n,n}$, respectively. Furthermore, $\zeta(\mathbf{K})$, $\rho(\mathbf{K})$, and $\xi(\mathbf{K})$ will denote the number of eigenvalues of $\mathbf{K} \in \mathbb{C}_{n,n}$ equal to, consecutively, zero, one, and belonging to the set $(0, 1)$.

The key role in the present paper is played by the notion of a projector. Hereafter, the symbol $\mathbb{C}_n^{\mathsf{P}}$ will mean the set of oblique projectors in $\mathbb{C}_{n,1}$ (idempotent matrices of order n), i.e.,

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$$\mathbb{C}_n^{\mathsf{P}} = \{\mathbf{K} \in \mathbb{C}_{n,n}: \mathbf{K}^2 = \mathbf{K}\},$$

whereas the symbol $\mathbb{C}_n^{\mathsf{OP}}$ will denote an important subset of $\mathbb{C}_n^{\mathsf{P}}$ consisting of orthogonal projectors (Hermitian idempotent matrices), i.e.,

$$\mathbb{C}_n^{\mathsf{OP}} = \{\mathbf{K} \in \mathbb{C}_{n,n}: \mathbf{K}^2 = \mathbf{K} = \mathbf{K}^*\}.$$

It is known that a given $\mathbf{P} \in \mathbb{C}_n^{\mathsf{P}}$ is a projector onto its column space $\mathscr{R}(\mathbf{P})$ along its null space $\mathscr{N}(\mathbf{P})$, where $\mathbb{C}_{n,1} = \mathscr{R}(\mathbf{P}) \oplus \mathscr{N}(\mathbf{P})$ with "$\oplus$" denoting the direct sum. Furthermore, an essential property of any orthogonal projector is that $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ if and only if it is expressible as $\mathbf{KK}^\dagger$ for some $\mathbf{K} \in \mathbb{C}_{n,m}$, where $\mathbf{K}^\dagger \in \mathbb{C}_{m,n}$ is the Moore–Penrose inverse of $\mathbf{K}$, i.e., the unique solution to the equations

$$\mathbf{KK}^\dagger\mathbf{K} = \mathbf{K}, \quad \mathbf{K}^\dagger\mathbf{KK}^\dagger = \mathbf{K}^\dagger, \quad (\mathbf{KK}^\dagger)^* = \mathbf{KK}^\dagger, \quad (\mathbf{K}^\dagger\mathbf{K})^* = \mathbf{K}^\dagger\mathbf{K}. \tag{1.1}$$

Then $\mathbf{P_K} = \mathbf{KK}^\dagger$ is the orthogonal projector onto $\mathscr{R}(\mathbf{K})$ and, consequently, $\mathbf{Q_K} = \mathbf{I}_n - \mathbf{KK}^\dagger$ is the orthogonal projector onto the orthogonal complement of $\mathscr{R}(\mathbf{K})$, denoted by $\mathscr{R}^\perp(\mathbf{K})$. Similarly, $\mathbf{P}_{\mathbf{K}^*} = \mathbf{K}^\dagger\mathbf{K}$ and $\mathbf{Q}_{\mathbf{K}^*} = \mathbf{I}_m - \mathbf{K}^\dagger\mathbf{K}$ are the orthogonal projectors onto $\mathscr{R}(\mathbf{K}^*)$ and $\mathscr{R}^\perp(\mathbf{K}^*)$, respectively.

Another matrix inverse which will be of interest in the present paper is the group inverse. Existence of such an inverse is restricted to square matrices only and for a given $\mathbf{K} \in \mathbb{C}_{n,n}$ it is the unique matrix $\mathbf{K}^\# \in \mathbb{C}_{n,n}$ satisfying the equations

$$\mathbf{KK}^\#\mathbf{K} = \mathbf{K}, \quad \mathbf{K}^\#\mathbf{KK}^\# = \mathbf{K}^\#, \quad \mathbf{KK}^\# = \mathbf{K}^\#\mathbf{K}. \tag{1.2}$$

It is known that not every square matrix has a group inverse and that the necessary and sufficient condition for a given matrix $\mathbf{K}$ to have such an inverse is that it is of index one, or, in other words, that $\mathrm{rk}(\mathbf{K}^2) = \mathrm{rk}(\mathbf{K})$.

Clearly, the Moore–Penrose inverse and the group inverse (if it exists) belong to the set of so called g-inverses, composed of matrices satisfying the first condition in (1.1) or (1.2). We will use the symbol $\mathbf{K}\{1\}$ to denote the set (always nonempty) of all g-inverses of $\mathbf{K} \in \mathbb{C}_{n,m}$, i.e.,

$$\mathbf{K}\{1\} = \{\mathbf{K}^- \in \mathbb{C}_{m,n}: \mathbf{KK}^-\mathbf{K} = \mathbf{K}\}. \tag{1.3}$$

Additional symbols used in what follows are $\mathbb{C}_n^{\mathsf{EP}}$ and $\mathbb{C}_n^{\mathsf{U}}$, denoting the subsets of $\mathbb{C}_{n,n}$ composed of range-Hermitian (also known as EP) and unitary matrices, respectively, i.e.,

$$\mathbb{C}_n^{\mathsf{EP}} = \{\mathbf{K} \in \mathbb{C}_{n,n}: \mathscr{R}(\mathbf{K}) = \mathscr{R}(\mathbf{K}^*)\} = \{\mathbf{K} \in \mathbb{C}_{n,n}: \mathbf{KK}^\dagger = \mathbf{K}^\dagger\mathbf{K}\},$$
$$\mathbb{C}_n^{\mathsf{U}} = \{\mathbf{K} \in \mathbb{C}_{n,n}: \mathbf{KK}^* = \mathbf{I}_n = \mathbf{K}^*\mathbf{K}\}.$$

From the point of view of the present paper, a crucial role is played by the known fact that $\mathbf{K} \in \mathbb{C}_n^{\mathsf{P}}$ if and only if it is expressible as $(\mathbf{PQ})^\dagger$, where $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$; see e.g. Lemma 2.3 in [19] or Corollary in [10]. Thus, $(\mathbf{PQ})^\dagger$ can be represented in the form

$$(\mathbf{PQ})^\dagger = \mathbf{V}\begin{pmatrix} \mathbf{I}_r & \mathbf{H} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^*, \tag{1.4}$$

where $r = \mathrm{rk}[(\mathbf{PQ})^\dagger]$, $\mathbf{V} \in \mathbb{C}_n^{\mathsf{U}}$, and $\mathbf{H} \in \mathbb{C}_{r,n-r}$; see [25, Theorem 5]. (Parenthetically note that, clearly, $\mathrm{rk}[(\mathbf{PQ})^\dagger] = \mathrm{tr}[(\mathbf{PQ})^\dagger]$, which is the well-known necessary condition for the idempotency of $(\mathbf{PQ})^\dagger$.) Consequently, by direct verification of definition (1.1), it follows that the product $\mathbf{PQ} = [(\mathbf{PQ})^\dagger]^\dagger$ can be expressed as

$$\mathbf{PQ} = \mathbf{V}\begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{F} & \mathbf{0} \end{pmatrix}\mathbf{V}^*, \tag{1.5}$$

with

$$\mathbf{E} = (\mathbf{I}_r + \mathbf{HH}^*)^{-1} \quad \text{and} \quad \mathbf{F} = \mathbf{H}^*\mathbf{E}. \tag{1.6}$$

In the present paper, several results involving product $\mathbf{PQ}$ are established by exploiting representation (1.5). These results concern such crucial notions as, for instance, rank, trace, range, null space, generalized inverses, and spectral properties of the product and its various functions. Particular attention is paid to alternative expressions for the requirement that $\mathbf{PQ} \in \mathbb{C}_n^{\mathrm{OP}}$, and these characterizations refer to such known classes of matrices as Hermitian, involutory, normal, star-dagger, unitary as well as partial isometries and semi-orthogonal projectors. Moreover, some results dealing with the notions of parallel sum and spectral norm are obtained. The variety of problems considered shows that representation (1.5) provides a powerful tool of wide applicability. Actually, the present paper is to some extent related to the paper by Groß [11], who proposed another (though actually equivalent to (1.5)) representation of the product of two orthogonal projectors, and within its frames provided a collection of inspiring results. However, the approach utilized in [11] is relatively complicated, for, instead of four submatrices occurring in (1.5), in the corresponding representation (2.1) in [11], nine submatrices are to be dealt with.

The lemma below provides several characteristics of submatrices $\mathbf{H}$, $\mathbf{E}$, and $\mathbf{F}$ occurring in (1.4) and (1.5), which will be useful in further considerations.

Lemma 1. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *with* $\mathbf{PQ}$ *having representation* (1.5). *Then:*

(i) $\mathbf{E}^2 + \mathbf{F}^*\mathbf{F} = \mathbf{E}$,
(ii) $\mathbf{I}_r - \mathbf{E} = \mathbf{EHH}^* = \mathbf{F}^*\mathbf{H}^* = \mathbf{HH}^*\mathbf{E} = \mathbf{HF}$,
(iii) $\mathbf{P_H E} = \mathbf{E P_H}$,
(iv) $\mathbf{Q_H E} = \mathbf{Q_H} = \mathbf{E Q_H}$,
(v) $\mathbf{F}^\dagger = \mathbf{E}^{-1}(\mathbf{H}^\dagger)^* = (\mathbf{H}^\dagger)^* + \mathbf{H}$,
(vi) $\mathbf{P_H} = \mathbf{F}^\dagger\mathbf{F}$, $\mathbf{P_{H^*}} = \mathbf{FF}^\dagger$,
(vii) $(\mathbf{I}_r - \mathbf{E})(\mathbf{I}_r - \mathbf{E})^\dagger = \mathbf{P_H} = (\mathbf{I}_r - \mathbf{E})^\dagger(\mathbf{I}_r - \mathbf{E})$,
(viii) $\mathbf{FH} + \mathbf{G} = \mathbf{I}_{n-r}$, *where* $\mathbf{G} = (\mathbf{I}_{n-r} + \mathbf{H}^*\mathbf{H})^{-1} = \mathbf{I}_{n-r} - \mathbf{H}^*\mathbf{EH}$,
(ix) $\mathscr{R}(\mathbf{I}_r - \mathbf{E}) = \mathscr{R}(\mathbf{F}^*)$,
(x) $\mathrm{tr}[(\mathbf{PQ})^\dagger] = \mathrm{rk}(\mathbf{PQ}) = r = \mathrm{rk}(\mathbf{E})$.

Proof. For the proof of point (i) observe that in view of (1.6), the following equalities are satisfied

$$\mathbf{E}^2 + \mathbf{F}^*\mathbf{F} = \mathbf{E}^2 + \mathbf{EHH}^*\mathbf{E} = \mathbf{E}(\mathbf{I}_r + \mathbf{HH}^*)\mathbf{E} = \mathbf{EE}^{-1}\mathbf{E} = \mathbf{E}.$$

The proof of point (ii) is based on the observation that from the left-hand side formula in (1.6) it follows that $\mathbf{EHH}^* = \mathbf{I}_r - \mathbf{E} = \mathbf{HH}^*\mathbf{E}$. Hence, taking into account the right-hand side formula in (1.6), we obtain $\mathbf{F}^*\mathbf{H}^* = \mathbf{I}_r - \mathbf{E} = \mathbf{HF}$.

To establish point (iii) note that, in view of the nonsingularity of $\mathbf{E}$, $\mathbf{P_H E} = \mathbf{E P_H} \Leftrightarrow \mathbf{E}^{-1}\mathbf{P_H} = \mathbf{P_H}\mathbf{E}^{-1}$. Hence, the assertion follows by utilizing the left-hand side formula in (1.6) and taking into account that (1.1) ensures that $\mathbf{HH}^*\mathbf{HH}^\dagger = \mathbf{HH}^*$.

Since point (iv) is obtained similarly as (iii), next we consider points (v) and (vi). Direct verifications of (1.1), with the use of the equality constituting point (iii), show that $\mathbf{F}^\dagger = \mathbf{E}^{-1}(\mathbf{H}^\dagger)^*$ is indeed the Moore–Penrose inverse of $\mathbf{F} = \mathbf{H}^*\mathbf{E}$, with $\mathbf{FF}^\dagger = \mathbf{H}^\dagger\mathbf{H}$ and $\mathbf{F}^\dagger\mathbf{F} = \mathbf{HH}^\dagger$. Note that these conditions constitute point (vi). The right-hand side equality in point (v) follows easily by replacing $\mathbf{E}^{-1}$ by $\mathbf{I}_r + \mathbf{HH}^*$.

For the proof of point (vii) first observe that, in view of the right-hand side formula in (1.6), it follows that

$$\mathscr{R}(\mathbf{FF}^*\mathbf{H}^*) = \mathscr{R}(\mathbf{H}^*\mathbf{EF}^*\mathbf{H}^*) \subseteq \mathscr{R}(\mathbf{H}^*), \tag{1.7}$$

and, taking into account the property $\mathbf{EHH}^* = \mathbf{HH}^*\mathbf{E}$ being a part of point (ii), that

$$\mathscr{R}(\mathbf{H}^*\mathbf{HF}) = \mathscr{R}(\mathbf{H}^*\mathbf{HH}^*\mathbf{E}) = \mathscr{R}(\mathbf{H}^*\mathbf{EHH}^*) \subseteq \mathscr{R}(\mathbf{F}). \tag{1.8}$$

Relationships (1.7) and (1.8) ensure that $(\mathbf{HF})^\dagger = \mathbf{F}^\dagger\mathbf{H}^\dagger$; see e.g. [9, Chapter 4, Ex. 22]. Hence, utilizing points (ii), (v), the left-hand side formula in (1.6), and the properties of the Moore–Penrose inverse, we get

$$\begin{aligned}(\mathbf{I}_r - \mathbf{E})(\mathbf{I}_r - \mathbf{E})^\dagger &= (\mathbf{I}_r - \mathbf{E})(\mathbf{HF})^\dagger = (\mathbf{I}_r - \mathbf{E})\mathbf{F}^\dagger\mathbf{H}^\dagger = (\mathbf{I}_r - \mathbf{E})\mathbf{E}^{-1}(\mathbf{H}^\dagger)^*\mathbf{H}^\dagger \\ &= (\mathbf{E}^{-1} - \mathbf{I}_r)(\mathbf{H}^\dagger)^*\mathbf{H}^\dagger = \mathbf{HH}^*(\mathbf{H}^\dagger)^*\mathbf{H}^\dagger = \mathbf{HH}^\dagger = \mathbf{P_H}.\end{aligned}$$

The right-hand side equality in (vii) is obtained analogously.

In the proof of point (viii), only equality $(\mathbf{I}_{n-r} + \mathbf{H}^*\mathbf{H})^{-1} = \mathbf{I}_{n-r} - \mathbf{H}^*\mathbf{E}\mathbf{H}$ needs to be justified, for if it is satisfied, then relationship $\mathbf{FH} + \mathbf{G} = \mathbf{I}_{n-r}$ follows easily by utilizing the right-hand side formula in (1.6). To see that both expressions for matrix $\mathbf{G}$ in point (viii) are equivalent, it suffices to refer to the so called Sherman–Morrison–Woodbury formula; see e.g. [18, p. 124].

The proof of point (ix) is based on the observation that point (vii) entails $\mathscr{R}(\mathbf{I}_r - \mathbf{E}) = \mathscr{R}(\mathbf{H})$. Hence, the assertion follows by combining point (vi) with the equivalence $\mathbf{HH}^\dagger = \mathbf{F}^\dagger\mathbf{F} \Leftrightarrow \mathscr{R}(\mathbf{H}) = \mathscr{R}(\mathbf{F}^*)$.

Finally, equalities constituting point (x) follow directly from (1.4) and (1.5) along with the left-hand side formula in (1.6). The proof is complete. □

The next section provides several characterizations of particular functions of $\mathbf{PQ}$, whereas Section 3 is devoted to results dealing with various inverses of $\mathbf{PQ}$ and its functions. The last section of the paper contains a collection of miscellaneous results. In all three sections one can find characterizations referring to an essential property of a pair of orthogonal projectors, being their commutativity, see e.g. [5]. It is known, for instance, that for $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$,

$$\mathbf{PQ} \in \mathbb{C}_n^{\mathrm{OP}} \Leftrightarrow \mathbf{PQ} = \mathbf{QP}. \tag{1.9}$$

From (1.5) it is seen that $\mathbf{PQ}$ is Hermitian if and only if $\mathbf{F} = \mathbf{0}$. In view of the right-hand side formula in (1.6), we get $\mathbf{F} = \mathbf{0} \Leftrightarrow \mathbf{H} = \mathbf{0}$, and, in consequence,

$$\mathbf{PQ} \in \mathbb{C}_n^{\mathrm{OP}} \Leftrightarrow \mathbf{H} = \mathbf{0}. \tag{1.10}$$

One of the tasks of the present paper is to establish counterparts of the equivalence (1.9), with the equality on the right-hand side replaced by other conditions involving $\mathbf{P}$ and $\mathbf{Q}$.

2. Particular functions of PQ

In what follows we provide several characterizations of particular functions of $\mathbf{PQ}$, including the product $\mathbf{PQ}$ itself, the sum $\mathbf{PQ} + \mathbf{QP}$, and differences $\mathbf{PQ} - \mathbf{QP}$, $\mathbf{I}_n - \mathbf{PQ}$. The first theorem concerns the spectral properties of the product $\mathbf{PQ}$. Its points (i) and (ii) constitute Lemma 2 in [11] and are recalled here for completeness; see also Theorem I in [1] and Lemma 2 in [4].

Theorem 1. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *with* $\mathbf{PQ}$ *having representation* (1.5). *Then:*

(i) *eigenvalues of* $\mathbf{PQ}$ *belong to the set* $[0, 1]$,
(ii) $\zeta(\mathbf{PQ}) = n - \mathrm{rk}(\mathbf{PQ})$,
(iii) $\rho(\mathbf{PQ}) = \mathrm{rk}(\mathbf{PQ}) - \mathrm{rk}(\mathbf{H})$,
(iv) $\xi(\mathbf{PQ}) = \mathrm{rk}(\mathbf{H})$.

Proof. Since statement (i) is known in the literature, and condition (ii) holds trivially, only the last two identities are to be shown. From (1.5) it follows that eigenvalues of $\mathbf{PQ}$ are either equal to zero or to the eigenvalues of $\mathbf{E}$. With $\lambda_j(.)$, $j = 1, \ldots, r$, denoting the jth eigenvalue of a matrix argument, from the left-hand side formula in (1.6), we get

$$\lambda_j(\mathbf{E}) = \lambda_j[(\mathbf{I}_r + \mathbf{HH}^*)^{-1}] = \frac{1}{1 + \lambda_j(\mathbf{HH}^*)}.$$

Hence, $\lambda_j(\mathbf{E}) = 1$ if and only if $\lambda_j(\mathbf{HH}^*) = 0$. In consequence, $\rho(\mathbf{E}) = r - \mathrm{rk}(\mathbf{H})$, from where point (iii) of the theorem follows.

Condition (iv) is obtained from the fact that $\zeta(\mathbf{PQ}) + \rho(\mathbf{PQ}) + \xi(\mathbf{PQ}) = n$ combined with points (ii) and (iii) of the theorem. □

Another consequences of (1.5) concern trace of $\mathbf{PQ}$. Namely, with the use of the left-hand side formula in (1.6), we have

$$\mathrm{tr}(\mathbf{PQ}) = \mathrm{tr}(\mathbf{E}) = \mathrm{tr}[(\mathbf{I}_r + \mathbf{HH}^*)^{-1}] = \sum_{j=1}^{r} \frac{1}{1+\lambda_j(\mathbf{HH}^*)}. \tag{2.1}$$

In view of the nonnegative definiteness of $\mathbf{HH}^*$, from (2.1) it is seen that $\mathrm{tr}(\mathbf{PQ}) \leqslant r = \mathrm{rk}(\mathbf{PQ})$. Furthermore, $\mathrm{tr}(\mathbf{PQ}) = r$ if and only if $\lambda_j(\mathbf{HH}^*) = 0$ for all js. This is attainable if and only if $\mathbf{H} = \mathbf{0}$, and thus we obtain equivalence $\mathbf{PQ} \in \mathbb{C}_n^{\mathrm{OP}} \Leftrightarrow \mathrm{tr}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{PQ})$. These facts were already given in statements (iv) and (v) of Corollary 1 in [11]. The next theorem provides yet another characterization of condition $\mathbf{PQ} \in \mathbb{C}_n^{\mathrm{OP}}$ referring to trace of $\mathbf{PQ}$.

Theorem 2. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$. *Then*

$$\rho(\mathbf{PQ}) \leqslant \mathrm{tr}(\mathbf{PQ}). \tag{2.2}$$

Moreover, $\rho(\mathbf{PQ}) = \mathrm{tr}(\mathbf{PQ})$ *if and only if* $\mathbf{PQ}$ *is an orthogonal projector.*

Proof. Inequality (2.2) is established on account of the observation that the following relationships hold

$$\mathrm{rk}(\mathbf{H}) = \mathrm{rk}(\mathbf{HH}^*) \geqslant \sum_{j=1}^{r} \frac{\lambda_j(\mathbf{HH}^*)}{1+\lambda_j(\mathbf{HH}^*)} = \sum_{j=1}^{r}\left(1 - \frac{1}{1+\lambda_j(\mathbf{HH}^*)}\right) = r - \mathrm{tr}(\mathbf{E}).$$

Hence, on account of point (iii) of Theorem 1 combined with the fact that $\mathrm{tr}(\mathbf{E}) = \mathrm{tr}(\mathbf{PQ})$, it is seen that inequality (2.2) necessarily holds. Moreover, inequality sign in (2.2) is replaced by an equality sign if and only if $\mathbf{H} = \mathbf{0}$, or, in other words, $\mathbf{PQ} \in \mathbb{C}_n^{\mathrm{OP}}$. $\square$

As mentioned in Section 1, every idempotent matrix is a projector onto its column space along its null space. According to a remark given in [10, p. 830], matrix $(\mathbf{PQ})^\dagger$ projects onto $\mathscr{R}[(\mathbf{PQ})^\dagger] = \mathscr{R}(\mathbf{QP})$. Since $\mathscr{N}[(\mathbf{PQ})^\dagger] = \mathscr{N}[(\mathbf{PQ})^*] = \mathscr{N}(\mathbf{QP})$, it is clear that $(\mathbf{PQ})^\dagger$ projects along $\mathscr{N}(\mathbf{QP})$. Similarly, $(\mathbf{QP})^\dagger$ projects onto $\mathscr{R}(\mathbf{PQ})$ along $\mathscr{N}(\mathbf{PQ})$. The next theorem provides an expression for the Moore–Penrose inverse of $(\mathbf{PQ})^\dagger$ involving the orthogonal projectors onto column spaces of $\mathbf{PQ}$ and $\mathbf{QP}$.

Theorem 3. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$. *Then* $\mathbf{P}_{\mathbf{PQ}}\mathbf{P}_{\mathbf{QP}} = \mathbf{PQ}$.

Proof. Since $(\mathbf{PQ})^\dagger$ is the projector onto $\mathscr{R}(\mathbf{QP})$ along $\mathscr{N}(\mathbf{QP})$, it follows that $(\mathbf{PQ})^\dagger\mathbf{QP} = \mathbf{QP}$. Hence,

$$\mathbf{P}_{\mathbf{PQ}}\mathbf{P}_{\mathbf{QP}} = \mathbf{PQQP}(\mathbf{QP})^\dagger = \mathbf{PQ}[\mathbf{QP}(\mathbf{QP})^\dagger]^* = \mathbf{PQ},$$

and the proof is complete. $\square$

Theorem 3 is supplemented by an observation that the following equivalences hold

$$(\mathbf{PQ})^\dagger \in \mathbb{C}_n^{\mathrm{OP}} \Leftrightarrow \mathbf{PQ} \in \mathbb{C}_n^{\mathrm{OP}} \Leftrightarrow \mathbf{PQ} \in \mathbb{C}_n^{\mathrm{EP}} \Leftrightarrow \mathscr{N}(\mathbf{QP}) = \mathscr{R}^\perp(\mathbf{QP});$$

see [18, p. 408].

In what follows we investigate the properties of the sum $\mathbf{PQ} + \mathbf{QP}$, being a Hermitian matrix of the form

$$\mathbf{PQ} + \mathbf{QP} = \mathbf{V}\begin{pmatrix} 2\mathbf{E} & \mathbf{F}^* \\ \mathbf{F} & \mathbf{0} \end{pmatrix}\mathbf{V}^*. \tag{2.3}$$

The first result concerns the rank.

Theorem 4. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$. *Then* $\mathrm{rk}(\mathbf{PQ} + \mathbf{QP}) = \mathrm{rk}(\mathbf{PQ}) + \xi(\mathbf{PQ})$.

Proof. Since $\mathscr{R}(\mathbf{F}^*) = \mathscr{R}(\mathbf{EH}) \subseteq \mathscr{R}(\mathbf{E})$, in view of the nonsingularity of $\mathbf{E}$, it follows from Corollary 19.1 in [17] that

$$\mathrm{rk}\begin{pmatrix}2\mathbf{E} & \mathbf{F}^*\\ \mathbf{F} & \mathbf{0}\end{pmatrix} = \mathrm{rk}(\mathbf{E}) + \mathrm{rk}(\mathbf{F}\mathbf{E}^{-1}\mathbf{F}^*) = r + \mathrm{rk}(\mathbf{H}^*\mathbf{E}\mathbf{H}) = r + \mathrm{rk}(\mathbf{H}).$$

Hence, the assertion follows on account of point (iv) of Theorem 1. □

The following two corollaries are obtained straightforwardly from Theorem 4.

Corollary 1. *Let* $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathrm{rk}(\mathbf{PQ}+\mathbf{QP}) = \mathrm{rk}(\mathbf{PQ})$ *if and only if* $\mathbf{PQ}$ *is an orthogonal projector.*

The next result concerns nonsingularity of $\mathbf{PQ}+\mathbf{QP}$.

Corollary 2. *Let* $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *with* $\mathbf{PQ}$ *having representation* (1.5). *Then the following statements are equivalent:*

(i) $\mathbf{PQ}+\mathbf{QP}$ *is nonsingular,*
(ii) $\xi(\mathbf{PQ}) = \zeta(\mathbf{PQ})$,
(iii) $\mathbf{H}$ *is of full column rank.*

If $\mathbf{K} \in \mathbb{C}_n^{\mathsf{P}}$ is nonsingular, then $\mathbf{K} = \mathbf{I}_n$. In view of Corollary 2, it is of interest to ask whether matrix $\mathbf{PQ}+\mathbf{QP}$ can be equal to $\mathbf{I}_n$, or, more generally, whether $\mathbf{PQ}+\mathbf{QP}$ can be idempotent. The answer to this question constitutes a corollary to the theorem below.

Theorem 5. *Let* $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *with* $\mathbf{PQ}$ *having representation* (1.5). *Then* $\mathrm{tr}[(\mathbf{PQ}+\mathbf{QP})^2] = 2[\mathrm{tr}(\mathbf{E}^2) + \mathrm{tr}(\mathbf{E})]$.

Proof. From (2.3) it follows that

$$(\mathbf{PQ}+\mathbf{QP})^2 = \mathbf{V}\begin{pmatrix}4\mathbf{E}^2 + \mathbf{F}^*\mathbf{F} & 2\mathbf{E}\mathbf{F}^*\\ 2\mathbf{F}\mathbf{E} & \mathbf{F}\mathbf{F}^*\end{pmatrix}\mathbf{V}^*,$$

and hence $\mathrm{tr}[(\mathbf{PQ}+\mathbf{QP})^2] = 4\mathrm{tr}(\mathbf{E}^2) + 2\mathrm{tr}(\mathbf{F}^*\mathbf{F})$. Since, on account of point (i) of Lemma, we get $\mathrm{tr}(\mathbf{F}^*\mathbf{F}) = \mathrm{tr}(\mathbf{E}) - \mathrm{tr}(\mathbf{E}^2)$, the assertion is established. □

Corollary 3. *Let* $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathbf{PQ}+\mathbf{QP}$ *is idempotent only if* $\mathbf{PQ} = \mathbf{0} = \mathbf{QP}$.

Proof. Clearly, idempotency of $\mathbf{PQ}+\mathbf{QP}$ ensures that $\mathrm{tr}[(\mathbf{PQ}+\mathbf{QP})^2] = \mathrm{tr}(\mathbf{PQ}+\mathbf{QP})$. Hence, on account of Theorem 5 combined with $\mathrm{tr}(\mathbf{PQ}+\mathbf{QP}) = 2\mathrm{tr}(\mathbf{E})$, we obtain condition $\mathrm{tr}(\mathbf{E}^2) = 0$, which implies $\mathbf{PQ} = \mathbf{0}$ $(= \mathbf{QP})$. □

In what follows we consider the properties of the column space of the matrix $\mathbf{PQ}+\mathbf{QP}$. As will turn out in a sequel, the condition constituting the next theorem plays an essential role in considerations over so called parallel sum of $\mathbf{PQ}$ and $\mathbf{QP}$.

Theorem 6. *Let* $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathscr{R}(\mathbf{PQ}) \subseteq \mathscr{R}(\mathbf{PQ}+\mathbf{QP})$.

Proof. The condition for ranges given in the theorem can be equivalently expressed as $(\mathbf{PQ}+\mathbf{QP})(\mathbf{PQ}+\mathbf{QP})^\dagger\mathbf{PQ} = \mathbf{PQ}$, where, as can be confirmed by direct verification of definition (1.1), the Moore–Penrose inverse of $\mathbf{PQ}+\mathbf{QP}$ is of the form

$$(\mathbf{PQ}+\mathbf{QP})^\dagger = \mathbf{V}\begin{pmatrix}\frac{1}{2}\mathbf{Q}_{\mathbf{H}} & \mathbf{F}^\dagger\\ (\mathbf{F}^\dagger)^* & -2[\mathbf{P}_{\mathbf{H}^*} + (\mathbf{H}^*\mathbf{H})^\dagger]\end{pmatrix}\mathbf{V}^*. \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$(\mathbf{PQ}+\mathbf{QP})(\mathbf{PQ}+\mathbf{QP})^\dagger = \mathbf{V}\begin{pmatrix}\mathbf{E}\mathbf{Q}_{\mathbf{H}} + \mathbf{F}^*(\mathbf{F}^\dagger)^* & 2\mathbf{E}\mathbf{F}^\dagger - 2\mathbf{F}^*[\mathbf{P}_{\mathbf{H}^*} + (\mathbf{H}^*\mathbf{H})^\dagger]\\ \frac{1}{2}\mathbf{F}\mathbf{Q}_{\mathbf{H}} & \mathbf{F}\mathbf{F}^\dagger\end{pmatrix}\mathbf{V}^*, \tag{2.5}$$

where, on account of points (iv) and (vi) of Lemma, we have $\mathbf{E}\mathbf{Q_H} + \mathbf{F}^*(\mathbf{F}^\dagger)^* = \mathbf{I}_r$ and $\mathbf{F}\mathbf{Q_H} = \mathbf{0}$. Furthermore, referring to point (v) of Lemma and utilizing the right-hand side formula in (1.6), we obtain

$$2\mathbf{E}\mathbf{F}^\dagger - 2\mathbf{F}^*[\mathbf{P}_{\mathbf{H}^*} + (\mathbf{H}^*\mathbf{H})^\dagger] = 2(\mathbf{H}^\dagger)^* - 2\mathbf{E}[\mathbf{H} + \mathbf{H}(\mathbf{H}^*\mathbf{H})^\dagger].$$

Combining relationship $(\mathbf{H}^\dagger)^* = \mathbf{H}(\mathbf{H}^*\mathbf{H})^\dagger$ (see e.g. [21, p. 67] with point (v) of Lemma, leads to the conclusion that the upper right entry of the matrix on the right-hand side of (2.5) is equal to zero matrix. In consequence, (2.5) can be rewritten in the form

$$(\mathbf{PQ} + \mathbf{QP})(\mathbf{PQ} + \mathbf{QP})^\dagger = \mathbf{V}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathbf{H}^*} \end{pmatrix}\mathbf{V}^*. \tag{2.6}$$

Observation that, in view of condition (vi) of Lemma, $\mathbf{PQ}$ given in (1.5) is invariant with respect to premultiplication by matrix (2.6), establishes the assertion. □

In view of the inclusion $\mathscr{R}(\mathbf{K} + \mathbf{L}) \subseteq \mathscr{R}(\mathbf{K}) + \mathscr{R}(\mathbf{L})$, which holds for any $\mathbf{K}, \mathbf{L} \in \mathbb{C}_{m,n}$, a clear consequence of Theorem 6 is what follows.

Corollary 4. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathscr{R}(\mathbf{PQ}) + \mathscr{R}(\mathbf{QP}) = \mathscr{R}(\mathbf{PQ} + \mathbf{QP})$.

Subsequently, we consider the difference $\mathbf{PQ} - \mathbf{QP}$. From (1.5) it follows that this matrix is of the form

$$\mathbf{PQ} - \mathbf{QP} = \mathbf{V}\begin{pmatrix} \mathbf{0} & -\mathbf{F}^* \\ \mathbf{F} & \mathbf{0} \end{pmatrix}\mathbf{V}^*, \tag{2.7}$$

i.e., is so called skew-Hermitian, whereas its second power is given by

$$(\mathbf{PQ} - \mathbf{QP})^2 = \mathbf{V}\begin{pmatrix} -\mathbf{F}^*\mathbf{F} & \mathbf{0} \\ \mathbf{0} & -\mathbf{F}\mathbf{F}^* \end{pmatrix}\mathbf{V}^*, \tag{2.8}$$

i.e., is Hermitian. The theorem below provides characterizations of the nonsingularity and idempotency of $\mathbf{PQ} - \mathbf{QP}$.

Theorem 7. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then:*

(i) $\mathrm{rk}(\mathbf{PQ} - \mathbf{QP}) = 2\xi(\mathbf{PQ})$,
(ii) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{PQ} - \mathbf{QP} \in \mathbb{C}_n^{\mathsf{P}} \Leftrightarrow (\mathbf{PQ} - \mathbf{QP})^2 = \mathbf{0}$.

Proof. From (2.7) it is seen that $\mathrm{rk}(\mathbf{PQ} - \mathbf{QP}) = 2\mathrm{rk}(\mathbf{F})$, where $\mathrm{rk}(\mathbf{F}) = \mathrm{rk}(\mathbf{H})$. Hence, condition (i) is established on account of point (iv) of Theorem 1.

As already pointed out, $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{F} = \mathbf{0}$. In consequence, the equivalences constituting point (ii) of the theorem follow straightforwardly from (2.7) and (2.8). □

From point (i) of Theorem 7 it is seen that $\mathbf{PQ} - \mathbf{QP}$ is nonsingular if and only if $\xi(\mathbf{PQ}) = n/2$; for another characterization of this type see Corollary 5 in [11]. Combining part (i) ⇔ (ii) of Corollary 2 with point (i) of Theorem 7 and referring to Theorem 1, leads to the following.

Corollary 5. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathbf{PQ} - \mathbf{QP}$ *and* $\mathbf{PQ} + \mathbf{QP}$ *are simultaneously nonsingular only if* $\mathrm{rk}(\mathbf{PQ}) = n/2$.

Formula (2.13) in [24] states that for $\mathbf{K}, \mathbf{L} \in \mathbb{C}_n^{\mathsf{P}}$,

$$\mathrm{rk}(\mathbf{K} + \mathbf{L}) + \mathrm{rk}(\mathbf{KL} - \mathbf{LK}) = \mathrm{rk}(\mathbf{K} - \mathbf{L}) + \mathrm{rk}(\mathbf{KL} + \mathbf{LK}). \tag{2.9}$$

Combining this result with Theorem 4 and point (i) of Theorem 7 shows that for $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ with $\mathbf{PQ}$ having representation (1.5), we have

$$\mathrm{rk}(\mathbf{P}+\mathbf{Q}) - \mathrm{rk}(\mathbf{P}-\mathbf{Q}) = \mathrm{rk}(\mathbf{PQ}) - \xi(\mathbf{PQ}).$$

In view of $\xi(\mathbf{PQ}) \leqslant \mathrm{rk}(\mathbf{PQ})$, it is clear that $\mathrm{rk}(\mathbf{P}-\mathbf{Q}) \leqslant \mathrm{rk}(\mathbf{P}+\mathbf{Q})$ and $\mathrm{rk}(\mathbf{PQ}-\mathbf{QP}) \leqslant \mathrm{rk}(\mathbf{PQ}+\mathbf{QP})$. Actually, these inequalities hold also for $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{P}}$; see [23]. Further consequences of (2.9) are given in what follows.

Corollary 6. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then:*

(i) $\mathrm{rk}(\mathbf{P}+\mathbf{Q}) = \mathrm{rk}(\mathbf{P}-\mathbf{Q}) \Leftrightarrow \mathrm{rk}(\mathbf{PQ}+\mathbf{QP}) = \mathrm{rk}(\mathbf{PQ}-\mathbf{QP}) \Leftrightarrow \mathrm{rk}(\mathbf{PQ}) = \xi(\mathbf{PQ})$,
(ii) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathrm{rk}(\mathbf{P}+\mathbf{Q}) - \mathrm{rk}(\mathbf{P}-\mathbf{Q}) = \mathrm{rk}(\mathbf{PQ})$.

As easy to verify by direct calculations, the Moore–Penrose inverse of the difference $\mathbf{PQ}-\mathbf{QP}$ is of the form

$$(\mathbf{PQ}-\mathbf{QP})^{\dagger} = \mathbf{V}\begin{pmatrix} \mathbf{0} & \mathbf{F}^{\dagger} \\ -(\mathbf{F}^{\dagger})^{*} & \mathbf{0} \end{pmatrix}\mathbf{V}^{*}, \tag{2.10}$$

and from (1.5) and (2.10) it follows that $\mathbf{PQ}(\mathbf{PQ}-\mathbf{QP})^{\dagger}\mathbf{QP} = \mathbf{0}$.

Another matrix of interest in the present paper is the difference $\mathbf{I}_n - \mathbf{PQ}$, which on account of (1.5) is of the form

$$\mathbf{I}_n - \mathbf{PQ} = \mathbf{V}\begin{pmatrix} \mathbf{I}_r - \mathbf{E} & \mathbf{0} \\ -\mathbf{F} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^{*}. \tag{2.11}$$

As usual, first we provide a result referring to the rank.

Theorem 8. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = \zeta(\mathbf{PQ}) + \xi(\mathbf{PQ})$.

Proof. In view of points (vi) and (vii) of Lemma, it can be verified that the Moore–Penrose inverse of $\mathbf{I}_n - \mathbf{PQ}$ is of the form

$$(\mathbf{I}_n - \mathbf{PQ})^{\dagger} = \mathbf{V}\begin{pmatrix} (\mathbf{I}_r - \mathbf{E})^{\dagger} & \mathbf{0} \\ \mathbf{F}(\mathbf{I}_r - \mathbf{E})^{\dagger} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^{*}, \tag{2.12}$$

with

$$(\mathbf{I}_n - \mathbf{PQ})(\mathbf{I}_n - \mathbf{PQ})^{\dagger} = \mathbf{V}\begin{pmatrix} \mathbf{P_H} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^{*} = (\mathbf{I}_n - \mathbf{PQ})^{\dagger}(\mathbf{I}_n - \mathbf{PQ}). \tag{2.13}$$

Since $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = \mathrm{rk}[(\mathbf{I}_n - \mathbf{PQ})(\mathbf{I}_n - \mathbf{PQ})^{\dagger}]$, from (2.13) we obtain $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = n - r + \mathrm{rk}(\mathbf{H})$. Hence, the assertion follows by utilizing points (ii) and (iv) of Theorem 1; see also formula (2.17) in [24]. $\square$

The next result concerns nonsingularity of $\mathbf{I}_n - \mathbf{PQ}$ and is a direct consequence of Theorem 8.

Corollary 7. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *with* $\mathbf{PQ}$ *having representation* (1.5). *Then the following statements are equivalent:*

(i) $\mathbf{I}_n - \mathbf{PQ}$ *is nonsingular*,
(ii) $\xi(\mathbf{PQ}) = \mathrm{rk}(\mathbf{PQ})$,
(iii) $\mathbf{H}$ *is of full row rank*.

It is of interest to inquire when, if whenever, $\mathrm{rk}[(\mathbf{I}_n - \mathbf{PQ})^{\dagger}] = \mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^{\dagger}]$. As is seen from the following remark, the answer to this question is related to condition $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Remark 1. From (2.12) it is seen that $\mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^{\dagger}] = n - r + \mathrm{tr}[(\mathbf{I}_r - \mathbf{E})^{\dagger}]$, whereas from (2.13) we have $\mathrm{rk}[(\mathbf{I}_n - \mathbf{PQ})^{\dagger}] = n - r + \mathrm{rk}(\mathbf{H})$, and thus

$$\mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^{\dagger}] - \mathrm{rk}[(\mathbf{I}_n - \mathbf{PQ})^{\dagger}] = \mathrm{tr}[(\mathbf{I}_r - \mathbf{E})^{\dagger}] - \mathrm{rk}(\mathbf{H}). \tag{2.14}$$

On account of [9, Chapter 4, Ex. 22], from the right-hand side formula in (1.6) combined with point (ii) of Lemma, we get $(\mathbf{I}_r - \mathbf{E})^\dagger = (\mathbf{HH}^*)^\dagger\mathbf{E}^{-1}$. Since $\mathbf{E}^{-1} = \mathbf{I}_r + \mathbf{HH}^*$, it further follows that

$$(\mathbf{I}_r - \mathbf{E})^\dagger = (\mathbf{HH}^*)^\dagger + (\mathbf{HH}^*)^\dagger\mathbf{HH}^*. \tag{2.15}$$

Taking into account that the properties of the rank and trace of a matrix ensure that

$$\mathrm{rk}(\mathbf{H}) = \mathrm{rk}(\mathbf{HH}^*) = \mathrm{tr}[\mathbf{HH}^*(\mathbf{HH}^*)^\dagger] = \mathrm{tr}[(\mathbf{HH}^*)^\dagger\mathbf{HH}^*],$$

identity (2.15) entails $\mathrm{tr}[(\mathbf{I}_r - \mathbf{E})^\dagger] = \mathrm{tr}[(\mathbf{HH}^*)^\dagger] + \mathrm{rk}(\mathbf{H})$. In consequence, from (2.14) it is seen that $\mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^\dagger] = \mathrm{rk}[(\mathbf{I}_n - \mathbf{PQ})^\dagger]$ if and only if $\mathbf{H} = \mathbf{0}$, or, equivalently, $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$; see also point (v) of Corollary 1 in [11].

According to Theorem 4.5 in [2], relationship $\mathcal{N}(\mathbf{I}_n - \mathbf{QP}) = \mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$ is always satisfied. An equivalent version of this condition, namely $\mathcal{R}(\mathbf{I}_n - \mathbf{PQ}) = \mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})$, was given in point (i) of Corollary 2 in [11]. Thus, it is seen that

$$\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} = (\mathbf{I}_n - \mathbf{PQ})(\mathbf{I}_n - \mathbf{PQ})^\dagger \tag{2.16}$$

and

$$\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{I}_n - (\mathbf{I}_n - \mathbf{PQ})(\mathbf{I}_n - \mathbf{PQ})^\dagger \tag{2.17}$$

are the orthogonal projectors onto $\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})$ and $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$, respectively. Several alternative formulae for $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$ are given in Theorem 4 in [20], with

$$\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = 2\mathbf{P}(\mathbf{P} + \mathbf{Q})^\dagger\mathbf{Q} \tag{2.18}$$

provided also by Groß [11, Corollary 3]. From (2.13) it follows that $\mathbf{I}_n - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{EP}}$, what ensures that $(\mathbf{I}_n - \mathbf{PQ})^\dagger = (\mathbf{I}_n - \mathbf{PQ})^\#$; see Theorem 4 in [9, Chapter 4]. In consequence, on account of Ex. 7.10.16 in [18], for $m \in \mathbb{N}$ we have

$$\lim_{m\to\infty}(\mathbf{PQ})^m = \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}, \tag{2.19}$$

what was observed in [11, Corollary 3]. Within the approach utilized in the present paper, from (2.13) we get

$$\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{V}\begin{pmatrix}\mathbf{Q_H} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^*, \tag{2.20}$$

whence it is clear that

$$\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] = \mathrm{tr}(\mathbf{Q_H}) = r - \mathrm{tr}(\mathbf{P_H}) = \mathrm{rk}(\mathbf{PQ}) - \xi(\mathbf{PQ}).$$

Another relevant observation is that combining (2.17) with (2.18) gives

$$\mathbf{I}_n - (\mathbf{I}_n - \mathbf{PQ})(\mathbf{I}_n - \mathbf{PQ})^\dagger = 2\mathbf{P}(\mathbf{P} + \mathbf{Q})^\dagger\mathbf{Q}. \tag{2.21}$$

Formula (2.21) will be useful to proof the following theorem.

Theorem 9. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathbf{PQ}(\mathbf{PQ} + \mathbf{QP})^\dagger\mathbf{QP} = \mathbf{P}(\mathbf{P} + \mathbf{Q})^\dagger\mathbf{Q}$.

Proof. On account of points (iv) and (vi) of Lemma, formulae (1.5) and (2.4) yield

$$\mathbf{PQ}(\mathbf{PQ} + \mathbf{QP})^\dagger\mathbf{QP} = \mathbf{V}\begin{pmatrix}\frac{1}{2}\mathbf{Q_H} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^*. \tag{2.22}$$

On the other hand, substituting matrix given in (2.13) into (2.21) leads to matrix $\mathbf{P}(\mathbf{P} + \mathbf{Q})^\dagger\mathbf{Q}$ of the same form as the one on the right-hand side of (2.22). □

In a comment to Theorem 9 it is worth mentioning that $\mathbf{PQ}(\mathbf{PQ} + \mathbf{QP})^\dagger\mathbf{QP}$ is in fact the parallel sum of $\mathbf{PQ}$ and $\mathbf{QP}$, i.e.,

$$\mathbf{PQ} \boxplus \mathbf{QP} = \mathbf{V}\begin{pmatrix} \frac{1}{2}\mathbf{Q_H} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^*. \tag{2.23}$$

In general, parallel sum of two matrices does not necessarily exist, but the existence of $\mathbf{PQ} \boxplus \mathbf{QP}$ is ensured by Theorem 6; see [3] or [21, Section 10.1.6].

One of the consequences of Theorem 9 is relationship $\mathscr{R}(\mathbf{PQ}) \cap \mathscr{R}(\mathbf{QP}) = \mathscr{R}(\mathbf{P}) \cap \mathscr{R}(\mathbf{Q})$, which, as can be shown by referring to the properties of a column space, holds regardless whether $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal or oblique projectors.

Considerations concerning the difference $\mathbf{I}_n - \mathbf{PQ}$ are concluded by a theorem providing several conditions equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$. One of the characterizations given therein refers to the set of star-dagger matrices, defined according to

$$\mathbb{C}_n^{\mathsf{SD}} = \{\mathbf{K} \in \mathbb{C}_{n,n}: \mathbf{K}^*\mathbf{K}^\dagger = \mathbf{K}^\dagger\mathbf{K}^*\}.$$

Theorem 10. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then the following statements are equivalent:*

(i) $\mathbf{PQ}$ *is an orthogonal projector,*
(ii) $\mathbf{I}_n - \mathbf{PQ}$ *is normal,*
(iii) $\mathbf{I}_n - \mathbf{PQ}$ *is a partial isometry,*
(iv) $\mathbf{I}_n - \mathbf{PQ}$ *is star-dagger,*
(v) $(\mathbf{I}_n - \mathbf{PQ})^\dagger$ *is idempotent,*
(vi) $(\mathbf{I}_n - \mathbf{PQ})^\dagger = \mathbf{I}_n - \mathbf{PQ}$,
(vii) $\mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^2] = \mathrm{tr}(\mathbf{I}_n - \mathbf{PQ})$,
(viii) $\mathrm{tr}(\mathbf{I}_n - \mathbf{PQ}) = \mathrm{rk}(\mathbf{I}_n - \mathbf{PQ})$.

Proof. For the proof of part (i) $\Leftrightarrow$ (ii) observe that $\mathbf{I}_n - \mathbf{PQ}$ is normal if and only if $\mathbf{PQP} = \mathbf{QPQ}$. Hence, the equivalence holds on account of Theorem in [8].

To establish part (i) $\Leftrightarrow$ (iii) simply note that from (2.11) and (2.12) it follows that $\mathbf{I}_n - \mathbf{PQ}$ is a partial isometry, i.e., satisfies $(\mathbf{I}_n - \mathbf{PQ})^\dagger = (\mathbf{I}_n - \mathbf{PQ})^*$ (see Theorem 5 in [9, Chapter 6]), if and only if $\mathbf{F} = \mathbf{0}$, or, in other words, $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Next we prove that $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{I}_n - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{SD}}$. On account of point (vii) of Lemma, from (2.11) and (2.12) it follows that

$$(\mathbf{I}_n - \mathbf{PQ})^*(\mathbf{I}_n - \mathbf{PQ})^\dagger = \mathbf{V}\begin{pmatrix} \mathbf{P_H} - \mathbf{F}^*\mathbf{F}(\mathbf{I}_r - \mathbf{E})^\dagger & -\mathbf{F}^* \\ \mathbf{F}(\mathbf{I}_r - \mathbf{E})^\dagger & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^*$$

and

$$(\mathbf{I}_n - \mathbf{PQ})^\dagger(\mathbf{I}_n - \mathbf{PQ})^* = \mathbf{V}\begin{pmatrix} \mathbf{P_H} & -(\mathbf{I}_r - \mathbf{E})^\dagger\mathbf{F}^* \\ \mathbf{FP_H} & -\mathbf{F}(\mathbf{I}_r - \mathbf{E})^\dagger\mathbf{F}^* + \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^*.$$

Hence, $\mathbf{I}_n - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{SD}}$ if and only if $\mathbf{F}^*\mathbf{F}(\mathbf{I}_r - \mathbf{E})^\dagger = \mathbf{0}$ and $\mathbf{F} = \mathbf{F}(\mathbf{I}_r - \mathbf{E})^\dagger$. Combining these conditions shows that $\mathbf{I}_n - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{SD}} \Rightarrow \mathbf{F} = \mathbf{0}$. The converse implication holds trivially.

To establish part (v) $\Rightarrow$ (i), note that from (2.12) it follows that $(\mathbf{I}_n - \mathbf{PQ})^\dagger \in \mathbb{C}_n^{\mathsf{P}}$ if and only if $(\mathbf{I}_r - \mathbf{E})^\dagger \in \mathbb{C}_n^{\mathsf{P}}$ and $\mathbf{F}[(\mathbf{I}_r - \mathbf{E})^\dagger]^2 = \mathbf{0}$. On account of point (vii) of Lemma, multiplying $[(\mathbf{I}_r - \mathbf{E})^\dagger]^2 = (\mathbf{I}_r - \mathbf{E})^\dagger$ by $\mathbf{I}_r - \mathbf{E}$ gives $\mathbf{P_H} = (\mathbf{I}_r - \mathbf{E})^\dagger$. Hence, it follows that $\mathbf{P_H} = \mathbf{I}_r - \mathbf{E}$, or, in other words, $\mathbf{Q_H} = \mathbf{E}$. Since $\mathbf{E}$ is nonsingular, we have $\mathbf{Q_H} = \mathbf{I}_r$, and, in consequence, $\mathbf{H} = \mathbf{0}$. Part (i) $\Rightarrow$ (v) is obvious.

Three equivalences left to be considered. To show that (vi) $\Rightarrow$ (i) observe that comparing (2.11) with (2.12) leads straightforwardly to conditions $\mathbf{I}_r - \mathbf{E} = (\mathbf{I}_r - \mathbf{E})^\dagger$ and $\mathbf{F} = -\mathbf{F}(\mathbf{I}_r - \mathbf{E})^\dagger$. Combining these two equalities gives $2\mathbf{F} = \mathbf{FE}$, and with the use of the left-hand side formula in (1.6) we further get $\mathbf{F} = -2\mathbf{FHH}^*$. On account of the right-hand side formula in (1.6), this condition can be rewritten in the form $\mathbf{H}^*\mathbf{E} = 2\mathbf{H}^*\mathbf{EHH}^*$, from where, in view of point (ii) of Lemma and the nonsingularity of $\mathbf{E}$, we get $\mathbf{HH}^* + 2\mathbf{HH}^*\mathbf{HH}^* = \mathbf{0}$. Hence, $\mathbf{H} = \mathbf{0}$. Furthermore, implication (i) $\Rightarrow$ (vi) also visibly holds.

Next we consider part (i) $\Leftrightarrow$ (vii). Clearly, $\mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^2] = \mathrm{tr}(\mathbf{I}_n - \mathbf{PQ})$ is equivalent to $\mathrm{tr}[(\mathbf{PQ})^2] = \mathrm{tr}(\mathbf{PQ})$, where

$$(\mathbf{PQ})^2 = \mathbf{V}\begin{pmatrix} \mathbf{E}^2 & \mathbf{0} \\ \mathbf{FE} & \mathbf{0} \end{pmatrix}\mathbf{V}^*. \tag{2.24}$$

Thus, $\mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^2] = \mathrm{tr}(\mathbf{I}_n - \mathbf{PQ}) \Leftrightarrow \mathrm{tr}(\mathbf{E}^2) = \mathrm{tr}(\mathbf{E})$. On account of point (i) of Lemma, the last equality is equivalent to $\mathrm{tr}(\mathbf{F}^*\mathbf{F}) = 0$, i.e., $\mathbf{F} = \mathbf{0}$.

Finally, from (2.11) it follows that $\mathrm{tr}(\mathbf{I}_n - \mathbf{PQ}) = n - \mathrm{tr}(\mathbf{E})$. Combining this condition with Theorem 8 shows that the equality constituting point (viii) is satisfied if and only if $\mathrm{tr}(\mathbf{E}) + \xi(\mathbf{PQ}) = \zeta(\mathbf{PQ}) - n$, or, equivalently,

$$\mathrm{tr}[(\mathbf{I}_r + \mathbf{HH}^*)^{-1}] + \mathrm{rk}(\mathbf{HH}^*) = r. \tag{2.25}$$

Let now $\mathscr{J}$ denote the set of those indexes j for which the jth eigenvalue of $\mathbf{HH}^*$ is nonzero, i.e.,

$$\mathscr{J} = \{j : j \in \{1, \ldots, r\} \text{ and } \lambda_j(\mathbf{HH}^*) > 0\}.$$

If $\overline{\mathscr{J}}$ is the set of indexes j for which $\lambda_j(\mathbf{HH}^*) = 0$, then

$$\mathrm{tr}[(\mathbf{I}_r + \mathbf{HH}^*)^{-1}] = \sum_{j=1}^{r} \frac{1}{1+\lambda_j} = \sum_{j\in\mathscr{J}} \frac{1}{1+\lambda_j} + \sum_{j\in\overline{\mathscr{J}}} \frac{1}{1+\lambda_j} = \sum_{j\in\mathscr{J}} \frac{1}{1+\lambda_j} + |\overline{\mathscr{J}}|,$$

where $\lambda_j = \lambda_j(\mathbf{HH}^*)$, $j = 1, \ldots, r$. Clearly, $|\mathscr{J}| = \mathrm{rk}(\mathbf{H}) = \mathrm{rk}(\mathbf{HH}^*)$, and thus (2.25) can be equivalently expressed as

$$\sum_{j\in\mathscr{J}} \frac{1}{1+\lambda_j} + |\overline{\mathscr{J}}| + |\mathscr{J}| = r,$$

what means that

$$\sum_{j\in\mathscr{J}} \frac{1}{1+\lambda_j} = 0.$$

However, this condition is satisfied only if $\mathscr{J} = \emptyset$ and, in consequence, we have $\mathbf{H} = \mathbf{0}$. (Note that equivalence (i) $\Leftrightarrow$ (viii) is a part of point (v) of Corollary 1 in [11].) $\square$

The present section is concluded by an observation that, unlike the difference $\mathbf{I}_n - \mathbf{PQ}$, the sum

$$\mathbf{I}_n + \mathbf{PQ} = \mathbf{V}\begin{pmatrix} \mathbf{I}_r + \mathbf{E} & \mathbf{0} \\ \mathbf{F} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^*$$

is necessarily nonsingular, with

$$(\mathbf{I}_n + \mathbf{PQ})^{-1} = \mathbf{V}\begin{pmatrix} (\mathbf{I}_r + \mathbf{E})^{-1} & \mathbf{0} \\ -\mathbf{F}(\mathbf{I}_r + \mathbf{E})^{-1} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^*.$$

3. Generalized inverses involving the product PQ

The next theorem provides several characterizations referring to the Moore–Penrose inverses of the products of $\mathbf{P}$ and $\mathbf{Q}$, and will be useful in further considerations.

Theorem 11. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then:*

(i) $(\mathbf{QPQ})^\dagger = (\mathbf{PQ})^\dagger(\mathbf{QP})^\dagger$,
(ii) $(\mathbf{PQ})^\dagger\mathbf{PQ} = \mathbf{QP}(\mathbf{QP})^\dagger$,
(iii) $(\mathbf{PQ})^\dagger = \mathbf{Q}(\mathbf{PQ})^\dagger$,
(iv) $(\mathbf{PQ})^\dagger = (\mathbf{PQ})^\dagger\mathbf{P}$,
(v) $\mathbf{PQ}(\mathbf{PQ})^\dagger = \mathbf{P}(\mathbf{PQ})^\dagger$,
(vi) $(\mathbf{QP})^\dagger\mathbf{QP} = (\mathbf{QP})^\dagger\mathbf{P}$.

Proof. Condition constituting point (i) follows straightforwardly from [9, Chapter 4, Ex. 22], whereas equality in point (ii) is satisfied trivially on account of the properties of the Moore–Penrose inverse. For the proofs of conditions (iii) and (iv) see the proof of Lemma 3 in [11]. Finally, the remaining two equalities are established on account of easily seen implications (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi). $\square$

A list of conditions equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$, each of which involves the Moore–Penrose inverse of the products of $\mathbf{P}$ and $\mathbf{Q}$, is provided in what follows.

Theorem 12. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then the following statements are equivalent:*

(i) $\mathbf{PQ}$ *is an orthogonal projector*,
(ii) $\mathbf{PQ}(\mathbf{PQ})^\dagger = \mathbf{PQP}$,
(iii) $(\mathbf{PQ})^\dagger = \mathbf{P}(\mathbf{PQ})^\dagger$,
(iv) $(\mathbf{PQ})^\dagger = \mathbf{Q}(\mathbf{QP})^\dagger$,
(v) $(\mathbf{PQ} - \mathbf{QP})^\dagger = (\mathbf{PQ})^\dagger - (\mathbf{QP})^\dagger$,
(vi) $(\mathbf{QPQ})^\dagger$ *is an orthogonal projector*,
(vii) $\mathrm{tr}[(\mathbf{PQP})^\dagger] = \mathrm{tr}[(\mathbf{PQ})^\dagger]$.

Proof. For the proof of the equivalence (i) $\Leftrightarrow$ (ii) observe that, in view of point (i) of Lemma, direct calculations with the use of (1.4) and (1.5) lead to the conclusion that condition (ii) is satisfied if and only if $\mathbf{E} \in \mathbb{C}_r^{\mathsf{P}}$, or, equivalently, $\mathbf{F} = \mathbf{0}$.

Regarding equivalences (i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv), only the sufficiency is to be established, for necessity is easily seen. To see that condition (iii) entails (i), observe that the following implications hold

$$(\mathbf{PQ})^\dagger = \mathbf{P}(\mathbf{PQ})^\dagger \Rightarrow \mathscr{R}[(\mathbf{PQ})^\dagger] \subseteq \mathscr{R}(\mathbf{P}) \Rightarrow \mathscr{R}(\mathbf{QP}) \subseteq \mathscr{R}(\mathbf{P}) \Rightarrow \mathbf{PQP} = \mathbf{QP}.$$

Hence, on account of Theorem in [8], it follows that $\mathbf{PQ} = \mathbf{QP}$ and, in consequence, $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Next, interchanging matrices $\mathbf{P}$ and $\mathbf{Q}$ in point (v) of Theorem 11 leads to $\mathbf{QP}(\mathbf{QP})^\dagger = \mathbf{Q}(\mathbf{QP})^\dagger$. Hence, identity in point (iv) of the theorem can be rewritten in the form $(\mathbf{PQ})^\dagger = \mathbf{QP}(\mathbf{QP})^\dagger$, showing that $(\mathbf{PQ})^\dagger \in \mathbb{C}_n^{\mathsf{OP}}$. In consequence, $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Further, from (1.4) it follows that

$$(\mathbf{PQ})^\dagger - (\mathbf{QP})^\dagger = \mathbf{V}\begin{pmatrix} \mathbf{0} & \mathbf{H} \\ -\mathbf{H}^* & \mathbf{0} \end{pmatrix}\mathbf{V}^*,$$

and, referring to (2.10), we obtain

$$(\mathbf{PQ} - \mathbf{QP})^\dagger = (\mathbf{PQ})^\dagger - (\mathbf{QP})^\dagger \Leftrightarrow \mathbf{H} = \mathbf{F}^\dagger. \tag{3.1}$$

Hence, in view of point (v) of Lemma, it is clear that the right-hand side equality in (3.1) is equivalent to $\mathbf{H} = \mathbf{0}$, what establishes part (i) $\Leftrightarrow$ (v) of the theorem.

To show that conditions (i) and (vi) are equivalent as well, first observe that, utilizing (1.4) and the left-hand side formula in (1.6), point (i) of Theorem 11 leads to

$$(\mathbf{QPQ})^\dagger = \mathbf{V}\begin{pmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^*. \tag{3.2}$$

Thus, $(\mathbf{QPQ})^\dagger \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{E} = \mathbf{I}_r$, or, in other words, $(\mathbf{QPQ})^\dagger \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{H} = \mathbf{0}$.

Finally, for the proof of part (i) $\Leftrightarrow$ (vii) notice that interchanging $\mathbf{P}$ and $\mathbf{Q}$ in point (i) of Theorem 11 leads to $(\mathbf{QP})^\dagger(\mathbf{PQ})^\dagger = (\mathbf{PQP})^\dagger$. Hence,

$$(\mathbf{PQP})^\dagger = \mathbf{V}\begin{pmatrix} \mathbf{I}_r & \mathbf{H} \\ \mathbf{H}^* & \mathbf{H}^*\mathbf{H} \end{pmatrix}\mathbf{V}^*, \tag{3.3}$$

and thus $\mathrm{tr}[(\mathbf{PQP})^\dagger] = r + \mathrm{tr}(\mathbf{H}^*\mathbf{H})$. Since from (1.4) it is seen that $\mathrm{tr}[(\mathbf{PQ})^\dagger] = r$, it is clear that equality constituting point (vii) is satisfied if and only if $\mathbf{H} = \mathbf{0}$. The proof is complete. $\square$

Unlike the Moore–Penrose inverse which exists for every matrix, the group inverse not necessarily does. Thus, it is natural to ask whether the group inverse of $\mathbf{PQ}$ always exists. The answer to this question is affirmative, and, by utilizing the present approach, this fact can be shown without much effort.

Theorem 13. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$ *with* $\mathbf{PQ}$ *having representation* (1.5) *and let* $m\in\mathbb{N}$. *Then*

$$(\mathbf{PQ})^m=\mathbf{V}\begin{pmatrix}\mathbf{E}^m & \mathbf{0}\\ \mathbf{H}^*\mathbf{E}^m & \mathbf{0}\end{pmatrix}\mathbf{V}^*.$$

Proof. The result is established by applying mathematical induction with reference to $\mathbf{PQ}$ of the form (1.5). □

Corollary 8. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$ *and let* $m\in\mathbb{N}$. *Then* $\mathrm{rk}[(\mathbf{PQ})^m]=\mathrm{rk}(\mathbf{PQ})$.

Remark 2. From Corollary 8 it follows that, in particular, $\mathrm{rk}[(\mathbf{PQ})^2]=\mathrm{rk}(\mathbf{PQ})$, i.e., matrix $\mathbf{PQ}$ is of index one, what is the necessary and sufficient condition for the existence of the group inverse of $\mathbf{PQ}$. From Lemma in [6] it is seen that the following equivalences are satisfied

$$\mathbb{C}_{n,1}=\mathscr{R}(\mathbf{PQ})\oplus\mathscr{N}(\mathbf{PQ})\Leftrightarrow\mathscr{R}(\mathbf{PQ})\cap\mathscr{N}(\mathbf{PQ})=\{\mathbf{0}\}\Leftrightarrow\dim[\mathscr{R}(\mathbf{PQ})+\mathscr{N}(\mathbf{PQ})]=n,$$

and each of the equalities occurring therein is equivalent to the claim that $(\mathbf{PQ})^{\#}$ exists; see also Solutions 29-5.2–29-5.5 [IMAGE – The Bulletin of the International Linear Algebra Society 30 (2003) p. 25] to IMAGE Problem 29-5 proposed by Groß and Trenkler [12] and [18, Ex. 5.10.12].

The representation of the group inverse of $\mathbf{PQ}$ given in (1.5) can be obtained by means of a full-rank factorization. Namely, let $\mathbf{K}\in\mathbb{C}_{n,r}$ and $\mathbf{L}\in\mathbb{C}_{r,n}$ be given by

$$\mathbf{K}=\mathbf{V}\begin{pmatrix}\mathbf{E}\\ \mathbf{F}\end{pmatrix}\quad\text{and}\quad\mathbf{L}=(\mathbf{I}_r:\mathbf{0})\,\mathbf{V}^*,$$

where $(\cdot:\cdot)$ denotes the columnwise partitioned matrix. Then, clearly, $\mathbf{KL}=\mathbf{PQ}$ and $\mathbf{LK}=\mathbf{E}$. From Theorem 3 in [9, Chapter 4] it follows that $(\mathbf{PQ})^{\#}=\mathbf{K}(\mathbf{LK})^{-2}\mathbf{L}$, and, in consequence,

$$(\mathbf{PQ})^{\#}=\mathbf{V}\begin{pmatrix}\mathbf{E}^{-1} & \mathbf{0}\\ \mathbf{H}^*\mathbf{E}^{-1} & \mathbf{0}\end{pmatrix}\mathbf{V}^*. \tag{3.4}$$

The next theorem provides two formulae for the group inverse of $\mathbf{PQ}$.

Theorem 14. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$. *Then:*

(i) $(\mathbf{PQ})^{\#}=(\mathbf{QP})^{\dagger}(\mathbf{PQ})^{\dagger}(\mathbf{QP})^{\dagger}$,
(ii) $(\mathbf{PQ})^{\#}=[(\mathbf{QP})^2]^{\dagger}$.

Proof. In view of the left-hand side formula in (1.6), condition (i) follows straightforwardly from (1.4) and (3.4).

For the proof of condition (ii) observe that (2.24) entails

$$(\mathbf{QP})^2=\mathbf{V}\begin{pmatrix}\mathbf{E}^2 & \mathbf{EF}^*\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^*, \tag{3.5}$$

and substituting this relationship along with (3.4) into definition (1.1), with the use of point (ii) of Lemma and the right-hand side formula in (1.6), shows that $(\mathbf{PQ})^{\#}$ is the Moore–Penrose inverse of $(\mathbf{QP})^2$. □

A theorem below provides characterizations of $\mathbf{PQ}\in\mathbb{C}_n^{\mathsf{OP}}$ involving the group inverse of $\mathbf{PQ}$.

Theorem 15. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$. *Then the following statements are equivalent:*

(i) $\mathbf{PQ}$ *is an orthogonal projector*,
(ii) $(\mathbf{PQ})^{\#}$ *is Hermitian*,
(iii) $(\mathbf{PQ})^{\#}$ *is idempotent*,

(iv) $\operatorname{tr}[(\mathbf{PQ})^{\#}] = \operatorname{rk}(\mathbf{PQ})$,
(v) $\mathbf{PQ}(\mathbf{PQ})^{\#} = \mathbf{QP}(\mathbf{PQ})^{\#}$,
(vi) $(\mathbf{I}_n - \mathbf{PQ})^{\#} = \mathbf{I}_n - (\mathbf{PQ})^{\#}$,
(vii) $\operatorname{tr}[\mathbf{I}_n - (\mathbf{PQ})^{\#}] = \operatorname{rk}[\mathbf{I}_n - (\mathbf{PQ})^{\#}]$.

Proof. From (3.4) it is seen that $(\mathbf{PQ})^{\#}$ is Hermitian if and only if $\mathbf{H} = \mathbf{0}$. Thus, part (i) $\Leftrightarrow$ (ii) is established. Further, from (3.4) it follows that

$$\operatorname{tr}[(\mathbf{PQ})^{\#}] = \operatorname{tr}(\mathbf{E}^{-1}) = \operatorname{tr}(\mathbf{I}_r + \mathbf{HH}^*) = r + \operatorname{tr}(\mathbf{HH}^*),$$

and it is clear that statements (i) or (ii) are equivalent to statement (iv). For the proof concerning statement (iii), observe that

$$[(\mathbf{PQ})^{\#}]^2 = \mathbf{V}\begin{pmatrix} \mathbf{E}^{-2} & \mathbf{0} \\ \mathbf{H}^*\mathbf{E}^{-2} & \mathbf{0} \end{pmatrix}\mathbf{V}^*$$

is equal to $(\mathbf{PQ})^{\#}$ if and only if $\mathbf{E} = \mathbf{I}_r$. It is thus seen that also for $(\mathbf{PQ})^{\#} \in \mathbb{C}_n^{\mathsf{P}}$ it is necessary and sufficient that $\mathbf{H} = \mathbf{0}$.

For the proof of the equivalences between condition (i) and the remaining three conditions given in the theorem, note that if $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$, then

$$\mathbf{PQ} = \mathbf{V}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^* = (\mathbf{PQ})^{\#}.$$

Thus, it is obvious that (i) $\Rightarrow$ (v), (vi), (vii).

To show that (v) $\Rightarrow$ (i) observe that from (1.5) and (3.4) it follows that equality constituting point (v) is satisfied only if $\mathbf{FE}^{-1} = \mathbf{0}$. Hence, $\mathbf{F} = \mathbf{0}$.

To establish part (vi) $\Rightarrow$ (i) note that, in view of the left-hand side formula in (1.6), from (3.4) we have

$$\mathbf{I}_n - (\mathbf{PQ})^{\#} = \mathbf{V}\begin{pmatrix} -\mathbf{HH}^* & \mathbf{0} \\ -\mathbf{H}^*\mathbf{E}^{-1} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{V}^*, \tag{3.6}$$

and, as already pointed out, $(\mathbf{I}_n - \mathbf{PQ})^{\#} = (\mathbf{I}_n - \mathbf{PQ})^{\dagger}$. Comparing (2.12) and (3.6) shows that condition (vi) of the theorem is satisfied if and only if $\mathbf{HH}^* = -(\mathbf{I}_r - \mathbf{E})^{\dagger}$ and $\mathbf{H}^*\mathbf{E}^{-1} = -\mathbf{F}(\mathbf{I}_r - \mathbf{E})^{\dagger}$. Hence, we get $\mathbf{H}^*\mathbf{E}^{-1} = \mathbf{FHH}^*$, from where, with the use of formulae (1.6), if follows that $\mathbf{H}^*(\mathbf{I}_r + \mathbf{HH}^*) = \mathbf{H}^*\mathbf{EHH}^*$. Further, applying point (i) of Lemma we obtain $\mathbf{H}^*(\mathbf{E} + \mathbf{HH}^*) = \mathbf{0}$. Since $\mathbf{E}$ is positive definite, the sum $\mathbf{E} + \mathbf{HH}^*$ is nonsingular. In consequence, we get $\mathbf{H}^* = \mathbf{0}$, what entails $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

In the final step of the proof consider point (vii). From (3.6) it is clear that

$$\operatorname{tr}[\mathbf{I}_n - (\mathbf{PQ})^{\#}] = n - r - \operatorname{tr}(\mathbf{HH}^*)$$

and

$$\operatorname{rk}[\mathbf{I}_n - (\mathbf{PQ})^{\#}] = n - r + \operatorname{rk}(\mathbf{HH}^*).$$

Hence, $\operatorname{tr}[\mathbf{I}_n - (\mathbf{PQ})^{\#}] = \operatorname{rk}[\mathbf{I}_n - (\mathbf{PQ})^{\#}]$ yields $\operatorname{rk}(\mathbf{HH}^*) = -\operatorname{tr}(\mathbf{HH}^*)$, and thus $\mathbf{H} = \mathbf{0}$. The proof is complete. $\square$

Unlike the Moore–Penrose inverse and group inverse, a g-inverse of a given matrix is in general not unique. Nevertheless, it is of interest to inquire about conditions ensuring that the product $\mathbf{Q}(\mathbf{PQ})^-\mathbf{P}$ is invariant with respect to the choice of $(\mathbf{PQ})^-$.

Theorem 16. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ and let $(\mathbf{PQ})^- \in \mathbf{PQ}\{1\}$. Then $\mathbf{Q}(\mathbf{PQ})^-\mathbf{P}$ is invariant with respect to the choice of $(\mathbf{PQ})^-$ if and only if*

$$\mathscr{R}(\mathbf{P}) \subseteq \mathscr{R}(\mathbf{PQ}) \quad \text{and} \quad \mathscr{R}(\mathbf{Q}) \subseteq \mathscr{R}(\mathbf{QP}). \tag{3.7}$$

Proof. The assertion follows directly from point (iii) of Lemma 2.2.4 in [21]. $\square$

Theorem 16 is supplemented by a particular version of a g-inverse of $\mathbf{PQ}$. Namely, as can be verified by referring to definition (1.3),

$$(\mathbf{PQ})^- = \mathbf{V}\begin{pmatrix}\mathbf{E}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^*$$

belongs to the set $\mathbf{PQ}\{1\}$.

An alternative version of conditions given in (3.7) can be obtained on account of Ex. 4.2.12 in [18], which ensures that

$$\mathscr{R}(\mathbf{P}) \subseteq \mathscr{R}(\mathbf{PQ}) \Leftrightarrow \mathscr{R}(\mathbf{P}) = \mathscr{R}(\mathbf{PQ}) \tag{3.8}$$

and, analogously,

$$\mathscr{R}(\mathbf{Q}) \subseteq \mathscr{R}(\mathbf{QP}) \Leftrightarrow \mathscr{R}(\mathbf{Q}) = \mathscr{R}(\mathbf{QP}). \tag{3.9}$$

Combining the equalities on the right-hand sides of equivalences (3.8) and (3.9) with Corollary 6.2 in [17] leads to the conclusion that $\mathbf{Q}(\mathbf{PQ})^-\mathbf{P}$ is invariant with respect to the choice of $(\mathbf{PQ})^-$ if and only if

$$\mathscr{R}(\mathbf{P}) \cap \mathscr{R}^\perp(\mathbf{Q}) = \{\mathbf{0}\} \quad \text{and} \quad \mathscr{R}(\mathbf{Q}) \cap \mathscr{R}^\perp(\mathbf{P}) = \{\mathbf{0}\},$$

or, equivalently,

$$\mathscr{R}(\mathbf{P}) \cap \mathscr{N}(\mathbf{Q}) = \{\mathbf{0}\} \quad \text{and} \quad \mathscr{R}(\mathbf{Q}) \cap \mathscr{N}(\mathbf{P}) = \{\mathbf{0}\}.$$

Another question of interest is when, if whenever, $(\mathbf{QP})^\dagger \in \mathbf{PQ}\{1\}$. It turns out that the necessary and sufficient condition ensuring it is $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Theorem 17. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. Then $(\mathbf{QP})^\dagger$ belongs to the set $\mathbf{PQ}\{1\}$ of all g-inverses of $\mathbf{PQ}$ if and only if $\mathbf{PQ}$ is an orthogonal projector.*

Proof. Direct calculations show that $(\mathbf{QP})^\dagger$, easily obtainable from (1.4), belongs to $\mathbf{PQ}\{1\}$ if and only if $\mathbf{E}^2 = \mathbf{E}$, or, equivalently, $\mathbf{E} = \mathbf{I}_r$. As already pointed out, another way to express this condition is $\mathbf{H} = \mathbf{0}$. $\square$

4. Miscellaneous results

The next theorem provides four further characterizations of $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Theorem 18. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. Then the following statements are equivalent:*

(i) $\mathbf{PQ}$ *is an orthogonal projector,*
(ii) $\mathrm{rk}(\mathbf{P}+\mathbf{Q}) + \mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{P}) + \mathrm{rk}(\mathbf{Q})$,
(iii) $\mathrm{rk}(2\mathbf{I}_n - \mathbf{P} - \mathbf{Q}) = \zeta(\mathbf{PQ})$,
(iv) $\mathrm{rk}[\mathbf{PQ} : \mathbf{QP}] = \mathrm{rk}(\mathbf{PQ})$,
(v) $\mathrm{rk}[\mathbf{I}_n - (\mathbf{PQ})^\dagger(\mathbf{QP})^\dagger] = \zeta(\mathbf{PQ})$,

where $(\cdot : \cdot)$ denotes the columnwise partitioned matrix.

Proof. According to Theorem 2.11 in [24], for $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{P}}$,

$$\mathrm{rk}(\mathbf{PQ}+\mathbf{QP}) = \mathrm{rk}(\mathbf{P}+\mathbf{Q}) + \mathrm{rk}(\mathbf{PQ}) + \mathrm{rk}(\mathbf{QP}) - \mathrm{rk}(\mathbf{P}) - \mathrm{rk}(\mathbf{Q}).$$

Combining this relationship with Theorem 4, yields

$$\xi(\mathbf{PQ}) = \mathrm{rk}(\mathbf{P}+\mathbf{Q}) + \mathrm{rk}(\mathbf{QP}) - \mathrm{rk}(\mathbf{P}) - \mathrm{rk}(\mathbf{Q}),$$

and hence, taking into account that $\mathrm{rk}(\mathbf{QP}) = \mathrm{rk}(\mathbf{PQ})$, the equivalence (i) $\Leftrightarrow$ (ii) is established.

Since the proof referring to statement (iii) follows directly from Corollary 2.17 in [24], next we consider statement (iv). In view of the properties of the rank of a matrix, we have

$$\mathrm{rk}[(\mathbf{PQ} : \mathbf{QP})] = \mathrm{rk}\left[(\mathbf{PQ} : \mathbf{QP})\begin{pmatrix}\mathbf{QP}\\ \mathbf{PQ}\end{pmatrix}\right] = \mathrm{rk}(\mathbf{PQP} + \mathbf{QPQ}),$$

where, utilizing (1.5),

$$\mathbf{PQP} + \mathbf{QPQ} = \mathbf{V}\begin{pmatrix}2\mathbf{E}^2 + \mathbf{F}^*\mathbf{F} & \mathbf{EF}^*\\ \mathbf{FE} & \mathbf{FF}^*\end{pmatrix}\mathbf{V}^*. \tag{4.1}$$

Point (i) of Lemma ensures that $2\mathbf{E}^2 + \mathbf{F}^*\mathbf{F} = \mathbf{E}(\mathbf{I}_r + \mathbf{E})$, and, since $\mathbf{E}$ is nonsingular, we can apply to (4.1) Corollary 19.1 in [17]. In consequence,

$$\mathrm{rk}[(\mathbf{PQ} : \mathbf{QP})] = \mathrm{rk}[\mathbf{E}(\mathbf{I}_r + \mathbf{E})] + \mathrm{rk}[\mathbf{FF}^* - \mathbf{FE}(\mathbf{I}_r + \mathbf{E})^{-1}\mathbf{F}^*].$$

Observing that, on the one hand, $\mathrm{rk}[\mathbf{E}(\mathbf{I}_r + \mathbf{E})] = r$, and, on the other hand, the following relationships are satisfied

$$\begin{aligned}\mathrm{rk}[\mathbf{FF}^* - \mathbf{FE}(\mathbf{I}_r + \mathbf{E})^{-1}\mathbf{F}^*] &= \mathrm{rk}\{\mathbf{F}[\mathbf{I}_r - \mathbf{E}(\mathbf{I}_r + \mathbf{E})^{-1}]\mathbf{F}^*\}\\ &= \mathrm{rk}\{\mathbf{F}[\mathbf{I}_r - (\mathbf{E} + \mathbf{I}_r - \mathbf{I}_r)(\mathbf{I}_r + \mathbf{E})^{-1}]\mathbf{F}^*\}\\ &= \mathrm{rk}[\mathbf{F}(\mathbf{I}_r + \mathbf{E})^{-1}\mathbf{F}^*] = \mathrm{rk}(\mathbf{F}) = (\mathbf{H}),\end{aligned}$$

we arrive at the assertion.

Finally, we establish the proof concerning point (v). From (1.4) it follows that

$$\mathbf{I}_n - (\mathbf{PQ})^\dagger(\mathbf{QP})^\dagger = \mathbf{V}\begin{pmatrix}-\mathbf{HH}^* & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n-r}\end{pmatrix}\mathbf{V}^*.$$

Hence, $\mathrm{rk}[\mathbf{I}_n - (\mathbf{PQ})^\dagger(\mathbf{QP})^\dagger] = n - r + \mathrm{rk}(\mathbf{H})$, and the equivalence (i) $\Leftrightarrow$ (v) is easily seen. (Parenthetically notice that, on account of point (i) of Theorem 11, $(\mathbf{PQ})^\dagger(\mathbf{QP})^\dagger = (\mathbf{QPQ})^\dagger$.) The proof is complete. $\square$

Further relationships concerning ranks are given in the following.

Theorem 19. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$. *Then:*

(i) $\mathrm{rk}[\mathbf{PQ}(\mathbf{I}_n - \mathbf{PQ})] = \xi(\mathbf{PQ})$,
(ii) $\mathrm{rk}(\mathbf{I}_n - \mathbf{QPQ}) = \zeta(\mathbf{PQ}) + \xi(\mathbf{PQ})$,
(iii) $\mathrm{rk}(\mathbf{QPQP}) = \mathrm{rk}(\mathbf{QPQ}) = \mathrm{rk}(\mathbf{QP})$.

Proof. For the proof of condition (i) observe that (1.5) and (2.11) entail

$$\mathbf{PQ}(\mathbf{I}_n - \mathbf{PQ}) = \mathbf{V}\begin{pmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{L} & \mathbf{0}\end{pmatrix}\mathbf{V}^*, \tag{4.2}$$

where $\mathbf{K} = \mathbf{E}(\mathbf{I}_r - \mathbf{E})$ and $\mathbf{L} = \mathbf{F}(\mathbf{I}_r - \mathbf{E})$. With such a notation, from [9, Chapter 4, Ex. 22] it follows that $[\mathbf{E}(\mathbf{I}_r - \mathbf{E})]^\dagger = (\mathbf{I}_r - \mathbf{E})^\dagger\mathbf{E}^{-1}$. Hence, on account of points (iii) and (vii) of Lemma, we get $\mathbf{KK}^\dagger = \mathbf{P}_\mathbf{H}$. Referring once again to Lemma, this time to its points (iii) and (vi), this further implies that $\mathbf{KK}^\dagger\mathbf{L}^* = \mathbf{L}^*$, or, equivalently, $\mathscr{R}(\mathbf{L}^*) \subseteq \mathscr{R}(\mathbf{K})$. This inclusion combined with Corollary 19.1 in [17] utilized with respect to (4.2), shows that $\mathrm{rk}[\mathbf{PQ}(\mathbf{I}_n - \mathbf{PQ})] = \mathrm{rk}(\mathbf{K})$. Since from $\mathbf{KK}^\dagger = \mathbf{P}_\mathbf{H}$ it follows that $\mathrm{rk}(\mathbf{K}) = \mathrm{rk}(\mathbf{H})$, the assertion is established.

In view of point (i) of Lemma, $\mathbf{PQ}$ given in (1.5) yields

$$\mathbf{I}_n - \mathbf{QPQ} = \mathbf{V}\begin{pmatrix}\mathbf{I}_r - \mathbf{E} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n-r}\end{pmatrix}\mathbf{V}^*.$$

Hence, $\mathrm{rk}(\mathbf{I}_n - \mathbf{QPQ}) = n - r + \mathrm{rk}(\mathbf{I}_r - \mathbf{E})$, and condition (ii) of the theorem is obtained on account of $\mathrm{rk}(\mathbf{I}_r - \mathbf{E}) = \mathrm{rk}(\mathbf{H})$, being a consequence of point (vii) of Lemma. Finally, the validity of the equalities constituting point (iii) of the theorem is obvious by comparing (1.5), (3.2), and (3.5). $\square$

Observe that point (iii) of Theorem 19 ensures existence of the group inverse of $\mathbf{QP}$. Furthermore, from Theorem 8 and condition (ii) of Theorem 19 it follows that $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = \mathrm{rk}(\mathbf{I}_n - \mathbf{QPQ})$. Another consequence of condition (ii) of Theorem 19 constitutes the corollary below.

Corollary 9. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathrm{rk}(\mathbf{I}_n - \mathbf{QPQ}) = \zeta(\mathbf{PQ})$ *if and only if* $\mathbf{PQ}$ *is an orthogonal projector.*

It is noteworthy that Corollary 9 corresponds to a particular version of point (i) of Theorem 10 in [25].

The next observation concerns Proposition 2.1 in [15]. If $\mathbf{A} = \mathbf{PQ}$ and $\mathbf{B} = \mathbf{QP}$, then condition $\mathbf{A}^*\mathbf{A} \leqslant_{\mathsf{RS}} \mathbf{A}^*\mathbf{B}$ given therein is equivalent to $\mathbf{QPQ} \leqslant_{\mathsf{RS}} \mathbf{QPQP}$, where $\leqslant_{\mathsf{RS}}$ denotes the rank subtractivity partial ordering introduced by Hartwig [14], and for $\mathbf{K}, \mathbf{L} \in \mathbb{C}_{m,n}$ defined as

$$\mathbf{K} \leqslant_{\mathsf{RS}} \mathbf{L} \quad \text{whenever } \mathrm{rk}(\mathbf{L} - \mathbf{K}) = \mathrm{rk}(\mathbf{L}) - \mathrm{rk}(\mathbf{K}).$$

Hence, it follows that

$$\mathbf{QPQ} \leqslant_{\mathsf{RS}} \mathbf{QPQP} \Leftrightarrow \mathrm{rk}(\mathbf{QPQP} - \mathbf{QPQ}) = \mathrm{rk}(\mathbf{QPQP}) - \mathrm{rk}(\mathbf{QPQ}),$$

and, in view of point (iii) of Theorem 19, we get

$$\mathbf{QPQ} \leqslant_{\mathsf{RS}} \mathbf{QPQP} \Leftrightarrow \mathbf{QPQ} = \mathbf{QPQP}.$$

Applying now Theorem in [8] leads to

$$\mathbf{QPQ} \leqslant_{\mathsf{RS}} \mathbf{QPQP} \Leftrightarrow \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}.$$

An interesting property of $\mathbf{E}$ defined by the left-hand side formula in (1.6) is given in what follows. Short and transparent proof of this result constitutes yet another example of the usefulness of the present approach.

Theorem 20. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *with* $\mathbf{PQ}$ *having representation* (1.5) *and let* $m \in \mathbb{N}$. *Then* $\lim_{m\to\infty} \mathbf{E}^m = \mathbf{Q_H}$.

Proof. Let $\mathrm{rk}(\mathbf{H}) = t$ and let

$$\mathbf{H} = \mathbf{R}\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{S}^* \tag{4.3}$$

be a singular value decomposition of matrix $\mathbf{H}$, with $\mathbf{R} \in \mathbb{C}_r^{\mathsf{U}}$, $\mathbf{S} \in \mathbb{C}_{n-r}^{\mathsf{U}}$, and $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_t)$, where $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_t > 0$ are singular values of $\mathbf{H}$; see [18, p. 412]. From (4.3) it follows that

$$\mathbf{E}^m = (\mathbf{I}_r + \mathbf{HH}^*)^{-m} = \mathbf{R}\begin{pmatrix} (\mathbf{I}_t + \boldsymbol{\Sigma}^2)^{-m} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{r-t} \end{pmatrix}\mathbf{R}^*$$

and

$$\mathbf{Q_H} = \mathbf{I}_r - \mathbf{HH}^\dagger = \mathbf{R}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{r-t} \end{pmatrix}\mathbf{R}^*.$$

In consequence, the assertion will be established if we show that

$$\lim_{m\to\infty} (\mathbf{I}_t + \boldsymbol{\Sigma}^2)^{-m} = \mathbf{0}. \tag{4.4}$$

Combining (2.19) and (2.20) with Theorem 13 leads to $\lim_{m\to\infty} \mathbf{H}^*\mathbf{E}^m = \mathbf{0}$, what entails $\lim_{m\to\infty} \boldsymbol{\Sigma}(\mathbf{I}_t + \boldsymbol{\Sigma}^2)^{-m} = \mathbf{0}$. Hence, it is seen that (4.4) holds and thus the proof is complete. □

The last five theorems deliver several further conditions equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$. The very next one deals with a full-rank factorization of $\mathbf{PQ}$. Suppose $\mathbf{K} \in \mathbb{C}_{n,r}$, $\mathbf{L} \in \mathbb{C}_{r,n}$ are such that $\mathrm{rk}(\mathbf{K}) = \mathrm{rk}(\mathbf{L}) = r$ and provide a full-rank factorization of $(\mathbf{PQ})^\dagger$, i.e., $(\mathbf{PQ})^\dagger = \mathbf{KL}$. Since $(\mathbf{PQ})^\dagger$ is idempotent, matrices $\mathbf{K}$ and $\mathbf{L}$ satisfy $\mathbf{LK} = \mathbf{I}_r$; see Lemma 2.2 in [16] or Lemma 2 in [9, Chapter 2]. On account of $\mathbf{PQ} = [(\mathbf{PQ})^\dagger]^\dagger$ it is seen that

$$\mathbf{PQ} = \mathbf{L}^\dagger \mathbf{K}^\dagger, \tag{4.5}$$

where $\mathbf{L}^\dagger = \mathbf{L}^*(\mathbf{LL}^*)^{-1}$ and $\mathbf{K}^\dagger = (\mathbf{K}^*\mathbf{K})^{-1}\mathbf{K}^*$. Observe that (4.5) is a full-rank factorization of $\mathbf{PQ}$.

Theorem 21. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ with $\mathbf{PQ}$ having a full-rank factorization (4.5). Then $\mathbf{K}^*\mathbf{KLL}^* = \mathbf{I}_r$ if and only if $\mathbf{PQ}$ is an orthogonal projector.*

Proof. According to Lemma 2.2 in [16] or Lemma 2 in [9, Chapter 2], product $\mathbf{PQ}$ of the form (4.5) is idempotent if and only if $\mathbf{K}^\dagger\mathbf{L}^\dagger = \mathbf{I}_r$, what can equivalently be expressed as $(\mathbf{K}^*\mathbf{K})^{-1}\mathbf{K}^*\mathbf{L}^*(\mathbf{LL}^*)^{-1} = \mathbf{I}_r$. Since $\mathbf{LK} = \mathbf{I}_r \Leftrightarrow \mathbf{K}^*\mathbf{L}^* = \mathbf{I}_r$, the assertion follows. $\square$

One of the characterizations given in the theorem below, refers to the notion of an involutory matrix, which is attributed to those square matrices whose second power is equal to the identity matrix. Observe that idempotency of $(\mathbf{PQ})^\dagger$ ensures that $2(\mathbf{PQ})^\dagger - \mathbf{I}_n$ is involutory.

Theorem 22. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. Then the following statements are equivalent:*

(i) $\mathbf{PQ}$ *is an orthogonal projector*,
(ii) $\mathrm{tr}[(\mathbf{PQ})^2] = \mathrm{tr}(\mathbf{PQ})$,
(iii) $2\mathbf{PQ} - \mathbf{I}_n$ *is involutory*,
(iv) $2\mathbf{PQ} - \mathbf{I}_n$ *is unitary*.

Proof. First observe that equivalence (i) $\Leftrightarrow$ (ii) is a direct consequence of the part (i) $\Leftrightarrow$ (vii) of Theorem 10.

Next, direct calculations show that $2\mathbf{PQ} - \mathbf{I}_n$ is involutory if and only if $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{P}}$ (parenthetically notice that this equivalence holds for any $\mathbf{P} \in \mathbb{C}_{n,m}$, $\mathbf{Q} \in \mathbb{C}_{m,n}$). Since $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$, from Theorem in [8] it is seen that part (i) $\Leftrightarrow$ (iii) is necessarily satisfied.

To establish equivalence between conditions (i) and (iv) observe that from the definition of the set $\mathbb{C}_n^{\mathsf{U}}$ it directly follows that $2\mathbf{PQ} - \mathbf{I}_n \in \mathbb{C}_n^{\mathsf{U}}$ if and only if $\mathbf{PQP} = \frac{1}{2}(\mathbf{PQ} + \mathbf{QP}) = \mathbf{QPQ}$. Hence, Theorem 2 in [7] leads to the assertion. $\square$

The next theorem shows that representation (1.5) provides a powerful tool to investigate also links between the classes of orthogonal and semi-orthogonal projectors. Recall that a semi-orthogonal projector is understood as a matrix $\mathbf{K} \in \mathbb{C}_{n,n}$ such that $\mathbf{K}^*\mathbf{K} = (\mathbf{K} + \mathbf{K}^*)/2$. The result below is related to Theorem 3 in [13].

Theorem 23. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. Then the following statements are equivalent*:

(i) $\mathbf{PQ}$ *is an orthogonal projector*,
(ii) $\mathbf{PQ}$ *is a semi-orthogonal projector*,
(iii) $(\mathbf{PQ})^\dagger$ *is a semi-orthogonal projector*,
(iv) $(\mathbf{PQ})^\#$ *is a semi-orthogonal projector*.

Proof. To show that (i) $\Leftrightarrow$ (ii) observe that, trivially, $\mathbf{PQ}$ is a semi-orthogonal projector if and only if $\mathbf{QPQ} = \frac{1}{2}(\mathbf{PQ} + \mathbf{QP})$. This condition occurred already in the proof of Theorem 22, as being equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ (or $2\mathbf{PQ} - \mathbf{I}_n \in \mathbb{C}_n^{\mathsf{U}}$).

The proof referring to statement (iii) follows by comparing $(\mathbf{QP})^\dagger(\mathbf{PQ})^\dagger$, being, on account of point (i) of Theorem 11, of the form (3.3), with $\frac{1}{2}[(\mathbf{PQ})^\dagger + (\mathbf{QP})^\dagger]$, given by

$$\frac{1}{2}[(\mathbf{PQ})^\dagger + (\mathbf{QP})^\dagger] = \mathbf{V}\begin{pmatrix} \mathbf{I}_r & \frac{1}{2}\mathbf{H} \\ \frac{1}{2}\mathbf{H}^* & \mathbf{0} \end{pmatrix}\mathbf{V}^*.$$

Hence, it is clear that $(\mathbf{PQ})^\dagger$ is a semi-orthogonal projector if and only if $\mathbf{H} = \mathbf{0}$, i.e., $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Finally, to establish part (i) $\Leftrightarrow$ (iv), we utilize (3.4) to compare

$$(\mathbf{QP})^{\#}(\mathbf{PQ})^{\#} = \mathbf{V}\begin{pmatrix}\mathbf{E}^{-2}+\mathbf{E}^{-1}\mathbf{HH}^*\mathbf{E}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{V}^*$$

with

$$\frac{1}{2}[(\mathbf{PQ})^{\#}+(\mathbf{QP})^{\#}] = \mathbf{V}\begin{pmatrix}\mathbf{E}^{-1} & \frac{1}{2}\mathbf{E}^{-1}\mathbf{H}\\ \frac{1}{2}\mathbf{H}^*\mathbf{E}^{-1} & \mathbf{0}\end{pmatrix}\mathbf{V}^*.$$

Hence, the equivalence between statement (iv) and $\mathbf{H}=\mathbf{0}$ is directly seen. $\square$

The next two theorems demonstrate the applicability of the present approach to consider the parallel sum of $\mathbf{PQ}$ and $\mathbf{QP}$. In the first of them we refer to the notion of a spectral norm of a matrix, defined for $\mathbf{K}\in\mathbb{C}_{m,n}$ to be the number $\|\mathbf{K}\| = \sqrt{\lambda_{\max}(\mathbf{K}^*\mathbf{K})}$, where $\lambda_{\max}(\cdot)$ is the maximal eigenvalue of a matrix argument. For scalars $\alpha,\beta\in\mathbb{R}$, we define $\alpha \boxplus \beta = \alpha\beta/(\alpha+\beta)$.

Theorem 24. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$. *Then*

$$\|\mathbf{PQ}\boxplus\mathbf{QP}\| \leqslant \|\mathbf{PQ}\|\boxplus\|\mathbf{QP}\|. \tag{4.6}$$

Moreover, $\|\mathbf{PQ}\boxplus\mathbf{QP}\| = \|\mathbf{PQ}\|\boxplus\|\mathbf{QP}\|$ *if and only if* $\mathbf{PQ}$ *is an orthogonal projector*.

Proof. Since the properties of the spectral norm ensure that $\|\mathbf{PQ}\| = \|\mathbf{QP}\|$, we have

$$\|\mathbf{PQ}\|\boxplus\|\mathbf{QP}\| = \frac{\|\mathbf{PQ}\|\|\mathbf{QP}\|}{\|\mathbf{PQ}\|+\|\mathbf{QP}\|} = \frac{1}{2}\|\mathbf{PQ}\|,$$

where, on account of (1.5), $\|\mathbf{PQ}\| = \sqrt{\lambda_{\max}(\mathbf{E})}$. On the other hand, from (2.23) it is seen that

$$\|\mathbf{PQ}\boxplus\mathbf{QP}\| = \left\|\begin{pmatrix}\frac{1}{2}\mathbf{Q}_{\mathbf{H}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\right\|,$$

whence (4.6) holds if and only if $1\leqslant\lambda_{\max}(\mathbf{E})$. However,

$$\lambda_{\max}(\mathbf{E}) = \max_j \frac{1}{1+\lambda_j(\mathbf{HH}^*)},\quad j=1,\ldots,r,$$

showing that inequality (4.6) is always satisfied. Moreover, inequality sign in (4.6) can be replaced by the equality sign if and only if $\max_j \frac{1}{1+\lambda_j(\mathbf{HH}^*)} = 1$, $j=1,\ldots,r$. This is equivalent to the claim that $\lambda_j(\mathbf{HH}^*)=0$ for all js, which in turn means that $\mathbf{H}=\mathbf{0}$. $\square$

The next theorem is a counterpart of the previous one obtained by replacing spectral norm by trace.

Theorem 25. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$. *Then*

$$\mathrm{tr}(\mathbf{PQ}\boxplus\mathbf{QP}) \leqslant \mathrm{tr}(\mathbf{PQ})\boxplus\mathrm{tr}(\mathbf{QP}). \tag{4.7}$$

Moreover, $\mathrm{tr}(\mathbf{PQ}\boxplus\mathbf{QP}) = \mathrm{tr}(\mathbf{PQ})\boxplus\mathrm{tr}(\mathbf{QP})$ *if and only if* $\mathbf{PQ}$ *is an orthogonal projector*.

Proof. On account of (1.5) and (2.23), inequality (4.7) can be equivalently expressed as $\mathrm{tr}(\mathbf{Q}_{\mathbf{H}})\leqslant\mathrm{tr}(\mathbf{E})$. However, from $\mathbf{Q}_{\mathbf{H}}=\mathbf{I}_r-\mathbf{P}_{\mathbf{H}}$ it follows that $\mathrm{tr}(\mathbf{Q}_{\mathbf{H}}) = r-\mathrm{rk}(\mathbf{H})$, and thus it is clear that (4.7) provides just another way of expressing inequality (2.2). In consequence, the proof of Theorem 2 is valid for the present one as well. $\square$

The paper is concluded be a remark demonstrating once again the usefulness of the approach utilized. It refers to the fact that any $\mathbf{K}\in\mathbb{C}_n^{\mathsf{P}}$, different from null and zero matrices, satisfies

$$\|\mathbf{K}\| = \|\mathbf{I}_n-\mathbf{K}\|. \tag{4.8}$$

Actually, identity (4.8) remains valid also when $\mathbf{K}$ is a projector onto an infinite dimensional Hilbert space. Its applicability in numerical analysis was pointed out by Szyld [22], who, besides providing some new proofs, recalled also several proofs of (4.8) scattered in the literature, some in simplified versions. Nevertheless, it seems that the justification of (4.8) obtained within our framework requires much less effort than any other of the known proofs.

Remark 3. Within the present formalism, identity (4.8) can be rewritten in the form

$$\|(\mathbf{PQ})^\dagger\| = \|\mathbf{I}_n - (\mathbf{PQ})^\dagger\|, \tag{4.9}$$

where $(\mathbf{PQ})^\dagger$ is defined in (1.4). Since, $\|(\mathbf{PQ})^\dagger\| = \|(\mathbf{QP})^\dagger\|$, on account of point (i) of Theorem 11, from (3.2) it follows that $\|(\mathbf{PQ})^\dagger\|^2 = \lambda_{\max}(\mathbf{E}^{-1})$. On the other hand, direct calculations with the use of (1.4), (3.3), and again point (i) of Theorem 11, show that

$$\|\mathbf{I}_n - (\mathbf{PQ})^\dagger\|^2 = \lambda_{\max}[\mathbf{I}_n - (\mathbf{PQ})^\dagger - (\mathbf{QP})^\dagger + (\mathbf{PQP})^\dagger],$$

where

$$\mathbf{I}_n - (\mathbf{PQ})^\dagger - (\mathbf{QP})^\dagger + (\mathbf{PQP})^\dagger = \mathbf{V}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{G}^{-1}\end{pmatrix}\mathbf{V}^*,$$

with $\mathbf{G} = (\mathbf{I}_{n-r} + \mathbf{H}^*\mathbf{H})^{-1}$. Since, $\mathbf{HH}^*$ and $\mathbf{H}^*\mathbf{H}$ have the same nonzero eigenvalues, the validity of (4.9), and thus also (4.8), is clear.

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